PHYS 1444 – Section 003
Lecture #20

Wednesday, Nov. 16, 2005
Dr. Jaehoon Yu

- Self Inductance
- Inductor
- Energy stored in a magnetic field
- LR circuit
- LC Circuit and EM Oscillation
- LRC circuit

Announcements

- Quiz Monday, Nov. 21 early in class
  - Covers: CH 29-4 to end of CH 30
- UTA Tech Fair today till 3pm
  - Lots of things to learn and lots of goodies
- A colloquium at 4pm this Wednesday
  - Dr. P. Nordlander from Rice University
  - About nano material and magnetic field they generate
  - Extra credit opportunity

Self Inductance

- The concept of inductance applies to a single isolated coil of N turns. How does this happen?
  - When a changing current passes through a coil
  - A changing magnetic flux is produced inside the coil
  - The changing magnetic flux in turn induces an emf in the same coil
  - This emf opposes the change in flux. Whose law is this?
    - Lenz's law
- What would this do?
  - If the current through the coil is increasing?
    - The increasing magnetic flux induces an emf that opposes the original current
    - This tends to impedes its increase, trying to maintain the original current
  - If the current through the coil is decreasing?
    - The decreasing flux induces an emf in the same direction as the current
    - This tends to increase the flux, trying to maintain the original current

Self Inductance

- Since the magnetic flux Φ_B passing through N turn coil is proportional to current I in the coil, we define self-inductance, $\mathcal{L}$:
  - $\boxed{L = \frac{N\Phi_B}{I}}$ Self Inductance
- The induced emf in a coil of self-inductance $\mathcal{L}$ is
  - $\varepsilon = -N\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$
  - What is the unit for self-inductance? $1H = 1V \cdot s/A = 1\Omega \cdot s$
- What does magnitude of $\mathcal{L}$ depend on?
  - Geometry and the presence of a ferromagnetic material
- Self inductance can be defined for any circuit or part of a circuit

So what in the world is the Inductance?

- It is an impediment onto the electrical current due to the existence of changing flux
- So?
- In other words, it behaves like a resistance to the varying current, like ac, that causes the constant change of flux
- But it also provides means to store energy, just like the capacitance

Inductor

- An electrical circuit always contain some inductance but is normally negligibly small
  - If a circuit contains a coil of many turns, it could have large inductance
- A coil that has significant inductance, $\mathcal{L}$, is called an inductor and is express with the symbol 
  - Precision resisters are normally wire wound
    - Would have both resistance and inductance
    - The inductance can be minimized by winding the wire back on itself in opposite direction to cancel magnetic flux
    - This is called a "non-inductive winding"
- If an inductor has negligible resistance, inductance controls a changing current
- For an AC current, the greater the inductance the less the ac current
  - An inductance thus acts something like a resistance to impede the flow of alternating current
  - The quality of an inductor is indicated by the term reactance or impedance

Example 30 – 3

Solenoid inductance. (a) Determine a formula for the self inductance $\mathcal{L}$ of a tightly wrapped solenoid (a long coil) containing N turns of wire in its length ℓ and whose cross-sectional area is A. (b) Calculate the value of $\mathcal{L}$ if N=100, ℓ=5.0cm, A=0.30cm^2 and the solenoid is air filled. (c) calculate $\mathcal{L}$ if the solenoid has an iron core with $\mu=4000\mu_0$.

What is the magnetic field inside a solenoid? $B = \mu_0 nI = \mu_0 NI/l$

The flux is, therefore, $\Phi_B = BA = \mu_0 NIA/l$

Using the formula for self inductance: $L = \frac{N\Phi_B}{I} = \frac{\mu_0 N^2 A}{l}$

(b) Using the formula above

$$L = \frac{\mu_0 N^2 A}{l} = \frac{\left(4\pi \times 10^{-7}\, T\cdot m/m\right)100^2\left(0.30\times 10^{-4}\, m^2\right)}{5.0\times 10^{-2}\, m} = 7.5\mu H$$

(c) The magnetic field with an iron core solenoid is $B = \mu NI/l$

$$L = \frac{\mu N^2 A}{l} = \frac{4000\left(4\pi \times 10^{-7}\, T\cdot m/m\right)100^2\left(0.30\times 10^{-4}\, m^2\right)}{5.0\times 10^{-2}\, m} = 0.030H = 30mH$$

Energy Stored in a Magnetic Field

- When an inductor of inductance $\mathcal{L}$ is carrying current I which is changing at a rate dI/dt, energy is supplied to the inductor at a rate
  - $P = I\varepsilon = LI\dfrac{dI}{dt}$
- What is the work needed to increase the current in an inductor from 0 to I?
  - The work dW done in a time dt is $dW = Pdt = LIdI$
  - Thus the total work needed to bring the current from 0 to I in an inductor is

$$W = \int dW = \int_0^I LIdI = L\left[\frac{1}{2}I^2\right]_0^I = \frac{1}{2}LI^2$$

Energy Stored in a Magnetic Field

- The work done to the system is the same as the energy stored in the inductor when it is carrying current I
  - $$\boxed{U = \frac{1}{2}LI^2}$$

    Energy Stored in a magnetic field inside an inductor
  - This compared to the energy stored in a capacitor, C, when the potential difference across it is V $U = \frac{1}{2}CV^2$
  - Just like the energy stored in a capacitor is considered to reside in the electric field between its plates
  - The energy in an inductor can be considered to be stored in its magnetic field

Stored Energy in terms of B

- So how is the stored energy written in terms of magnetic field B?
  - Inductance of an ideal solenoid without a fringe effect

    $$L = \mu_0 N^2 A/l$$

  - The magnetic field in a solenoid is $B = \mu_0 NI/l$
  - Thus the energy stored in an inductor is

    $$U = \frac{1}{2}LI^2 = \frac{1}{2}\frac{\mu_0 N^2 A}{l}\left(\frac{Bl}{\mu_0 N}\right)^2 = \frac{1}{2}\frac{B^2}{\mu_0}Al$$

    What is this?

    Volume V

    $$\boxed{U = \frac{1}{2}\frac{B^2}{\mu_0}Al}$$

    E

  - Thus the energy density is

    $$u = \frac{U}{V} = \frac{U}{Al} = \frac{1}{2}\frac{B^2}{\mu_0}$$

    $$\boxed{u = \frac{1}{2}\frac{B^2}{\mu_0}}$$

    E density

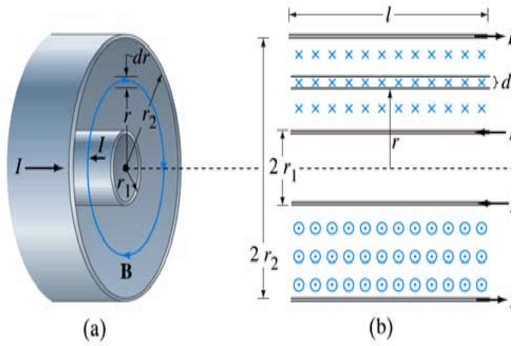
  - This formula is valid to any region of space
  - If a ferromagnetic material is present, μ_0 becomes μ.

What volume does Al represent?

The volume inside a solenoid!!

Example 30 – 5

Energy stored in a coaxial cable. (a) How much energy is being stored per unit length in a coaxial cable whose conductors have radii r_1 and r_2 and which carry a current I? (b) What is the energy density highest?



(a) The inductance per unit length for a coaxial cable is $$\frac{L}{l}=\frac{\mu_0}{2\pi}\ln\frac{r_2}{r_1}$$

Thus the energy stored per unit length is $$\frac{U}{l}=\frac{1}{2}\frac{LI^2}{l}=\frac{\mu_0 I^2}{4\pi}\ln\frac{r_2}{r_1}$$

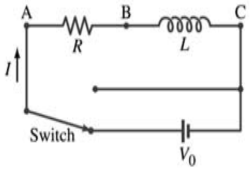
(b) Since the magnetic field is $$B=\frac{\mu_0 I}{2\pi r}$$

And the energy density is $$u=\frac{1}{2}\frac{B^2}{\mu_0}$$

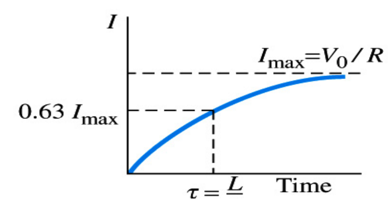
The energy density is highest where the B is highest. B is highest close to $r=r_1$, near the surface of the inner conductor.

LR Circuits

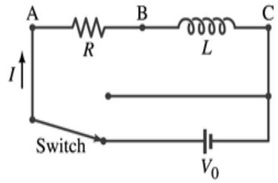
- What happens when an emf is applied to an inductor?
  - An inductor has some resistance, however negligible



    - So an inductor can be drawn as a circuit of separate resistance and coil. Do you what this kind of circuit is called? LR Circuit
  - At an instance the switch is thrown to apply an emf to the circuit
    - The current starts to flow, increasing from 0
    - This change is opposed by the induced emf in the inductor ➔ the emf at point B is higher than point C
    - However there is a voltage drop at the resistance which reduces the voltage across inductance
    - Thus the current increases less rapidly
    - The overall behavior of the current is gradual increase, reaching to the maximum current $I_{max}=V_0/R$.

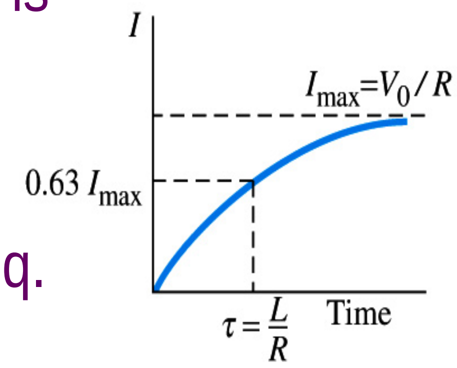


LR Circuits



- This can be shown w/ Kirchhoff rule loop rules
  - The emfs in the circuit are the battery voltage V0 and the emf ε=-$\mathcal{L}$(dI/dt) in the inductor opposing the increasing current
  - The sum of the potential changes around the loop is

$$V_0 + \varepsilon - IR = V_0 - L\,dI/dt - IR \quad 0$$

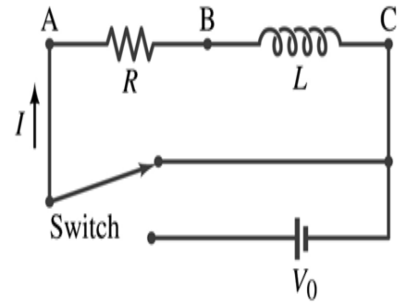


  - Where I is the current at any instant
  - By rearranging the terms we obtain a differential eq.
  - $L\,dI/dt + IR = V_0$
  - We can integrate just as in RC circuit $\int_{I=0}^{I} \frac{dI}{V_0 - IR} = \int_{t=0}^{t} \frac{dt}{L}$
  - So the solution is $-\frac{1}{R}\ln\left(\frac{V_0 - IR}{V_0}\right) = \frac{t}{L}$

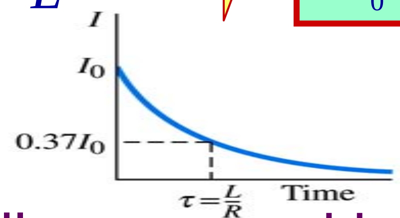
$$\boxed{I = V_0\left(1 - e^{-t/\tau}\right)/R = I_{max}\left(1 - e^{-t/\tau}\right)}$$

  - Where τ=L/R
    - This is the time constant τ of the LR circuit and is the time required for the current I to reach 0.63 of the maximum

Discharge of LR Circuits



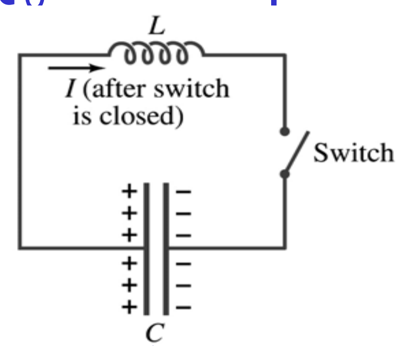
- If the switch is flipped away from the battery
  - The differential equation becomes
  - $L\,dI/dt + IR = 0$
  - So the integration is $\int_{I=0}^{I} \frac{dI}{IR} = \int_{t=0}^{t} \frac{dt}{L}$ $\Rightarrow$ $\boxed{\ln \frac{I}{I_0} = -\frac{R}{L}t}$
  - Which results in the solution
  - $\boxed{I = I_0 e^{-\frac{R}{L}t} = I_0 e^{-t/\tau}}$



  - The current decays exponentially to zero with the time constant τ=L/R
  - So there always is a reaction time when an system with a coil, such as an electromagnet, is turned on or off.
  - The current in LR circuit behaves almost the same as that in RC circuit but the time constant is inversely proportional to R in LR circuit unlike RC circuit

LC Circuit and EM Oscillations

- What's an LC circuit?
  - A circuit that contains only an inductor and a capacitor
    - How is this possible? There is no source of emf.
    - Well you can imagine the a circuit with a fully charged capacitor
    - In this circuit, we assume inductor does not have any resistance
- Let's assume that the capacitor originally has $+Q_0$ on one plate and $-Q_0$ on the other



  - Suppose the switch is closed at t=0
  - The capacitor starts discharge
  - The current flow through the inductor increases
  - Applying Kirchhoff's loop rule, we obtain $-L\,dI/dt + Q/C = 0$
  - Since the current flows out of the plate with positive charge, the charge on the plate reduces, so $I=-dQ/dt$. Thus the differential equation can be rewritten

$$\boxed{\frac{d^2Q}{dt} + \frac{Q}{LC} = 0}$$

LC Circuit and EM Oscillations

- This equation looks the same as hat of the harmonic oscillation
  - So the solution for this second order differential equation is
  - $Q = Q_0 \cos\left(\varpi t + \phi\right)$ Charge on the capacitor oscillates sinusoidally
  - Inserting the solution back into the differential equation
  - $\frac{d^2Q}{dt} + \frac{Q}{LC} = -\varpi^2 Q_0 \cos\left(\varpi t + \phi\right) + Q_0 \cos\left(\varpi t + \phi\right) / LC = 0$
  - Solving this equation for ω, we obtain $\varpi = 2\pi f = 1/\sqrt{LC}$
  - The current in the inductor is
  - $I = -dQ/dt = \varpi Q_0 \sin\left(\varpi t + \phi\right) = I_0 \sin\left(\varpi t + \phi\right)$
  - So the current also is sinusoidal w/ the maximum

$$I_0 = \varpi Q_0 = Q_0 / \sqrt{LC}$$

Energies in LC Circuit & EM Oscillation

- The energy stored in the electric field of the capacitor at any time t is $U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{Q_0^2}{2C}\cos^2(\varpi t + \phi)$
- The energy stored in the magnetic field in the inductor at the same instant is $U_B = \frac{1}{2}LI^2 = \frac{Q_0^2}{2C}\sin^2(\varpi t + \phi)$
- Thus, the total energy in LC circuit at any instant is

$$U = U_E + U_B = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2 = \frac{Q_0^2}{2C}\left[\cos^2(\varpi t + \phi) + \sin^2(\varpi t + \phi)\right] = \frac{Q_0^2}{2C}$$

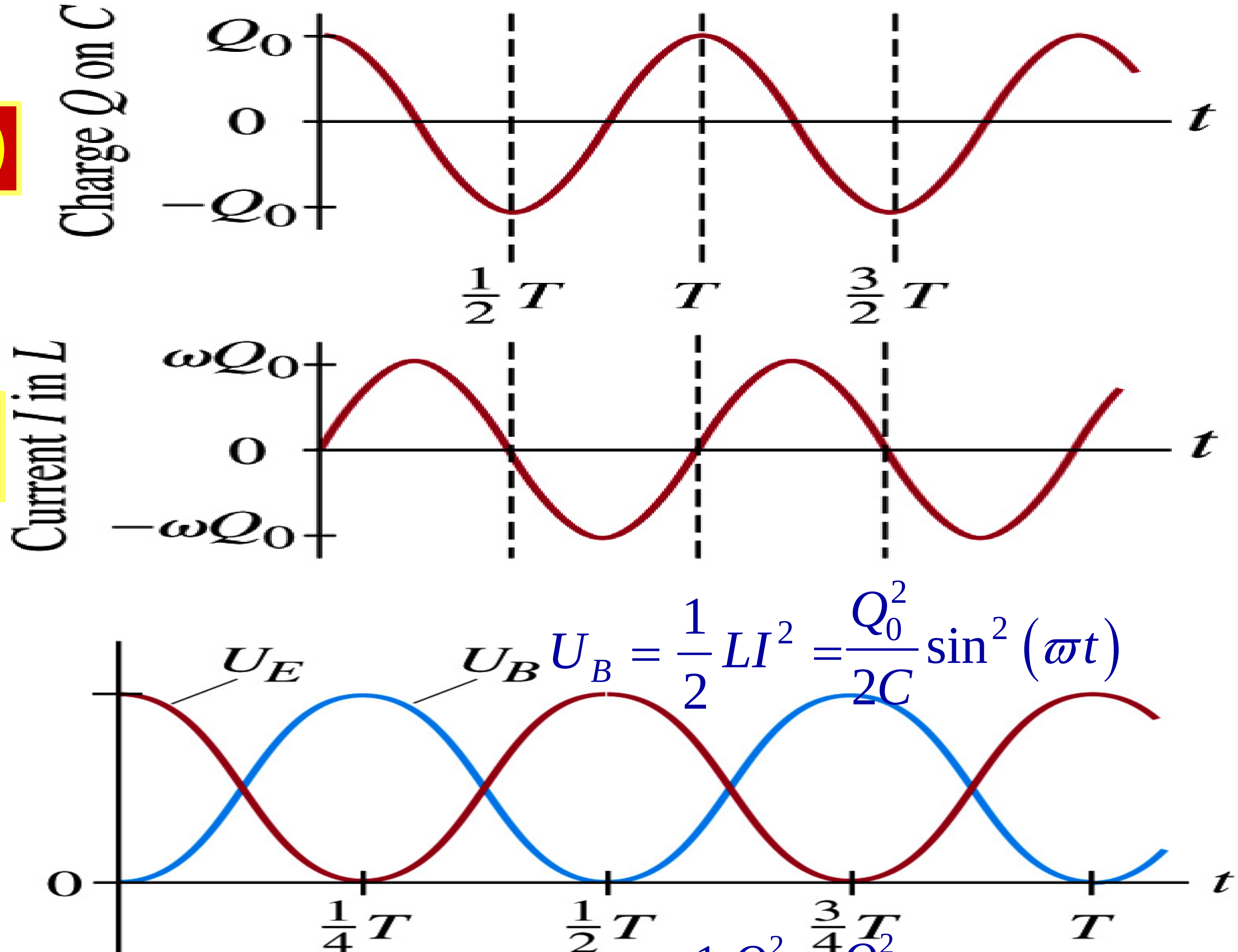
- So the total EM energy is constant and is conserved.
- This LC circuit is an LC oscillator or EM oscillation
  - The charge Q oscillates back and forth, from one plate of the capacitor to the other
  - The current also oscillates back and forth as well

LC Circuit Behaviors

$$Q = Q_0 \cos(\varpi t)$$

$$I = \varpi Q_0 \sin(\varpi t)$$

$$U = \frac{Q_0^2}{2C}$$



$$U_B = \frac{1}{2}LI^2 = \frac{Q_0^2}{2C}\sin^2(\varpi t)$$

$$U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{Q_0^2}{2C}\cos^2(\varpi t)$$

Example 30 – 7

LC Circuit. A 1200-pF capacitor is fully charged by a 500-V dc power supply. It is disconnected from the power supply and is connected, at t=0, to a 75-mH inductor. Determine: (a) The initial charge on the capacitor, (b) the maximum current, (c) the frequency f and period T of oscillation; and (d) the total energy oscillating in the system.

(a) The 500-V power supply, charges the capacitor to

$$Q = CV = \left(1200\times10^{-12}F\right)\cdot 500V = 6.0\times10^{-7}C$$

(b) The maximum current is

$$I_{\max} = \varpi Q_0 = \frac{Q_0}{\sqrt{LC}} = \frac{6.0\times10^{-7}C}{\sqrt{75\times10^{-3}H\times1.2\times10^{-9}F}} = 63mA$$

(c) The frequency is

$$f = \frac{\varpi}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{7.5\times10^{-3}H\cdot1.2\times10^{-9}F}} = 1.7\times10^{3}Hz$$

The period is

$$T = \frac{1}{f} = 6.0\times10^{-5}S$$

(d) The total energy in the system

$$U = \frac{Q_0^2}{2C} = \frac{\left(6.0\times10^{-7}C\right)^2}{2\cdot1.2\times10^{-9}F} = 1.5\times10^{-4}J$$

LC Oscillations w/ Resistance (LRC circuit)

- There is no such thing as zero resistance coil so all LC circuits have some resistance
  - So to be more realistic, the effect of the resistance should be taken into account
  - Suppose the capacitor is charged up to Q_0 initially and the switch is closed in the circuit at t=0
  - What do you expect to happen to the energy in the circuit?
    - Well, due to the resistance we expect some energy will be lost through the resister via a thermal conversion
  - What about the oscillation? Will it look the same as the ideal LC circuit we dealt with?
  - No? OK then how would be different?
  - The oscillation would be damped due to the energy loss.

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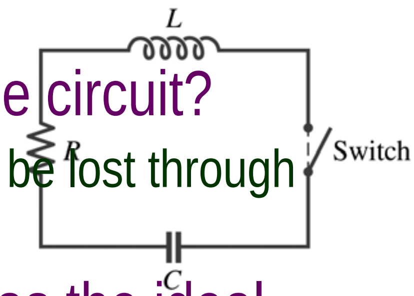
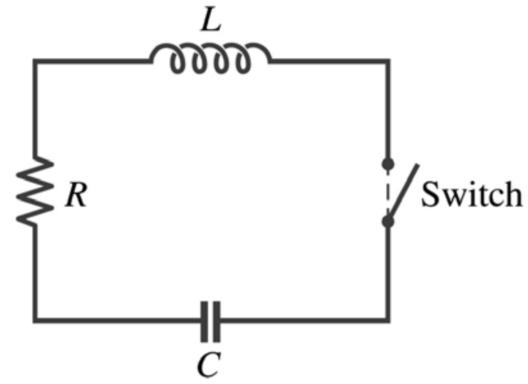


LC Oscillations w/ Resistance (LRC circuit)



- Now let's do some analysis
- From Kirchhoff's loop rule, we obtain

$$-L\,dI/dt - IR + \frac{Q}{C} = 0$$

- Since I=dQ/dt, the equation becomes

$$-L\frac{d^2Q}{dt} - R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

  – Which is identical to that of a damped oscillator

- The solution of the equation is $Q = Q_0 e^{-\frac{R}{2L}t}\cos\left(\varpi' t + \phi\right)$
  – Where the angular frequency is $\varpi' = \sqrt{1/LC - R^2/4L^2}$
    - R^2<4L/C: Underdamped
    - R^2>4L/C: Overdampled