1443-501 Spring 2002
Lecture #3

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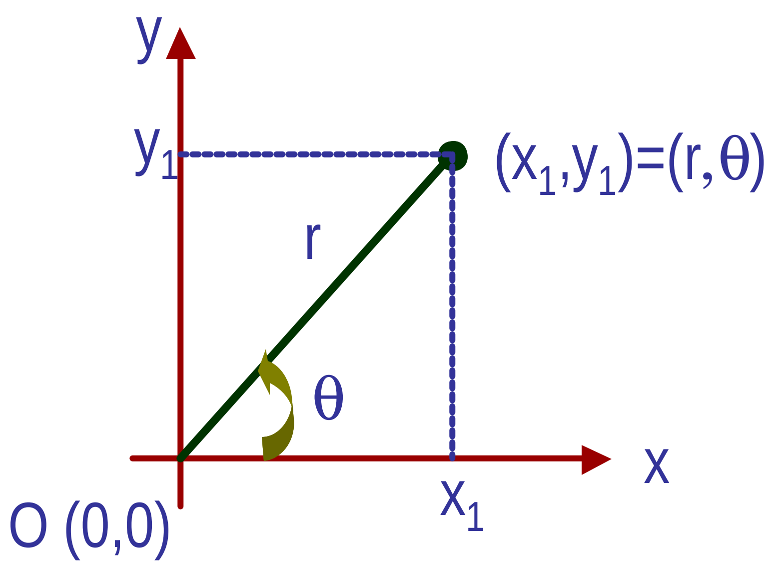
1. Coordinate Systems
2. Vector Properties and Operations
3. 2-dim Displacement, Velocity, & Acceleration
4. 2-dim Motion Under Constant Acceleration
5. Projectile Motion

Coordinate Systems

- Make it easy to express locations or positions
- Two commonly used systems, depending on convenience
  - Cartesian (Rectangular) Coordinate System
    - Coordinates are expressed in (x,y)
  - Polar Coordinate System
    - Coordinates are expressed in (r,θ)
- Vectors become a lot easier to express and compute



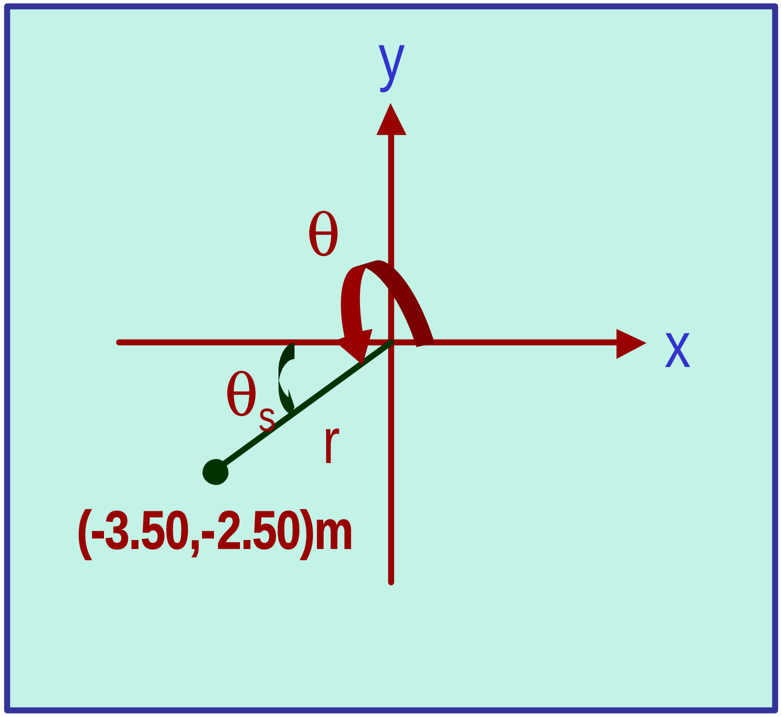
How are Cartesian and Polar coordinates related??

$$x = r\cos q$$
$$y = r\sin q$$

$$r = \sqrt{\left(x_1^2 + y_1^2\right)}$$
$$\tan q = \frac{y_1}{x_1}$$

Example 3.1

Cartesian Coordinate of a point in the xy plane are (x,y)= (-3.50,-2.50)m. Find the polar coordinates of this point.



$$r = \sqrt{\left(x_1^2 + y_1^2\right)} =$$

$$\sqrt{\left((-3.50)^2 + (-2.50)^2\right)}$$

$$= \sqrt{18.5} = 4.30(m)$$

$$q = 180 + q_s$$

$$\tan q_s = \frac{-2.50}{-3.50} = \frac{5}{7}$$

$$q_s = \tan^{-1}\left(\frac{5}{7}\right) = 35.5^\circ$$

$$\therefore q = 180 + q_s = 180^\circ + 35.5^\circ = 216^\circ$$

Vector and Scalar

Vector quantities have both magnitude (size) and direction

Force, gravitational pull, momentum

Normally denoted in **BOLD** letters, ***F***, or a letter with arrow on top $\vec{F}$

Their sizes or magnitudes are denoted with normal letters letters, *F*, or absolute values: $|\vec{F}|$ or $|F|$

Scalar quantities have magnitude only

Can be completely specified with a value and its unit

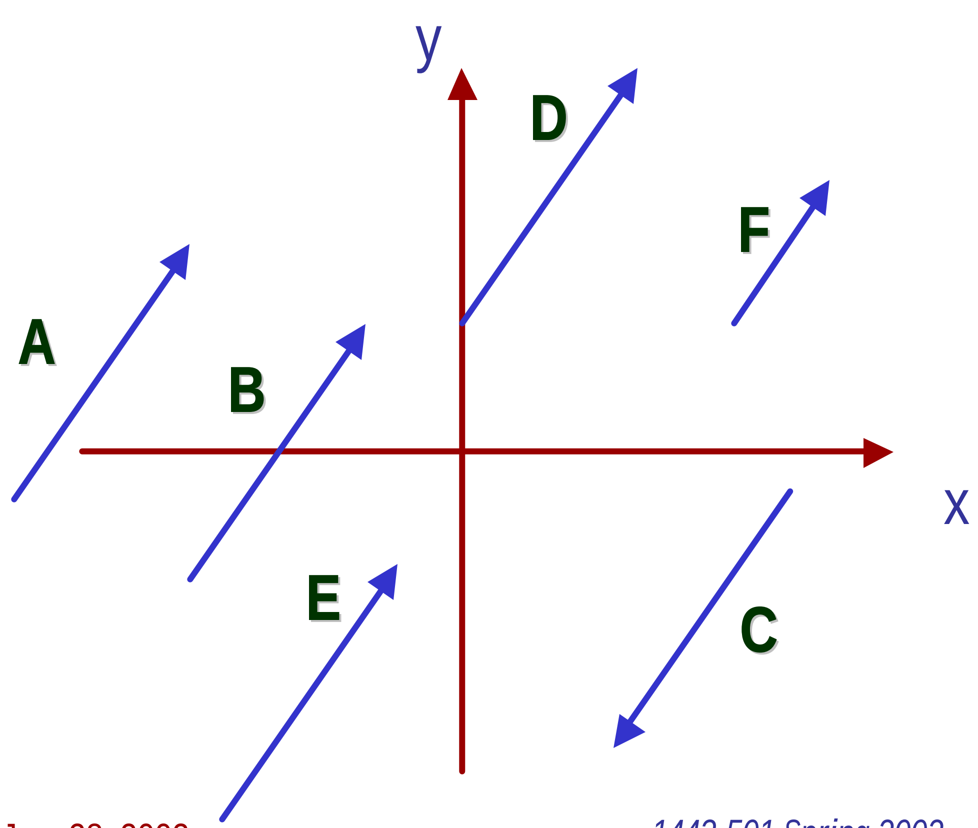
Energy, heat, mass, weight

Normally denoted in normal letters, *E*

Both have units!!!

Properties of Vectors

- Two vectors are the same if their sizes and the direction are the same, no matter where they are on a coordinate system.



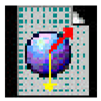
Which ones are the same vectors?

A=B=E=D

Why aren't the others?

C: The same magnitude but opposite direction:
C=-A: A negative vector

F: The same direction but different magnitude



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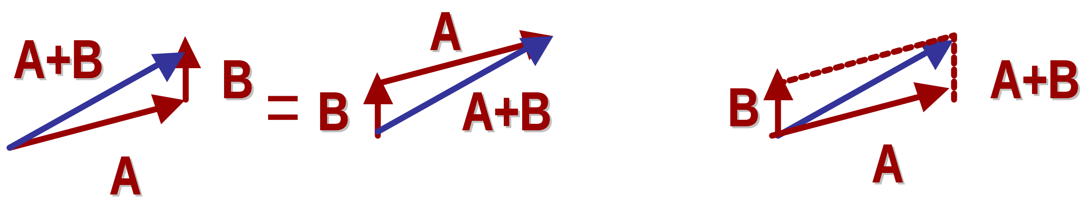
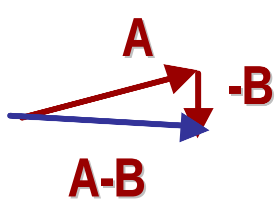
Vector Operations

- Addition:
  - Triangular Method: One can add vectors by connecting the head of one vector to the tail of the other
  - Parallelogram method: Connect the tails of the two vectors and extend
  - Addition is commutative: Changing order of operation does not affect the results **A+B=B+A**, **A+B+C+D+E=E+C+A+B+D**



- Subtraction:
  - The same as adding a negative vector: **A** - **B** = **A** + (-**B**)



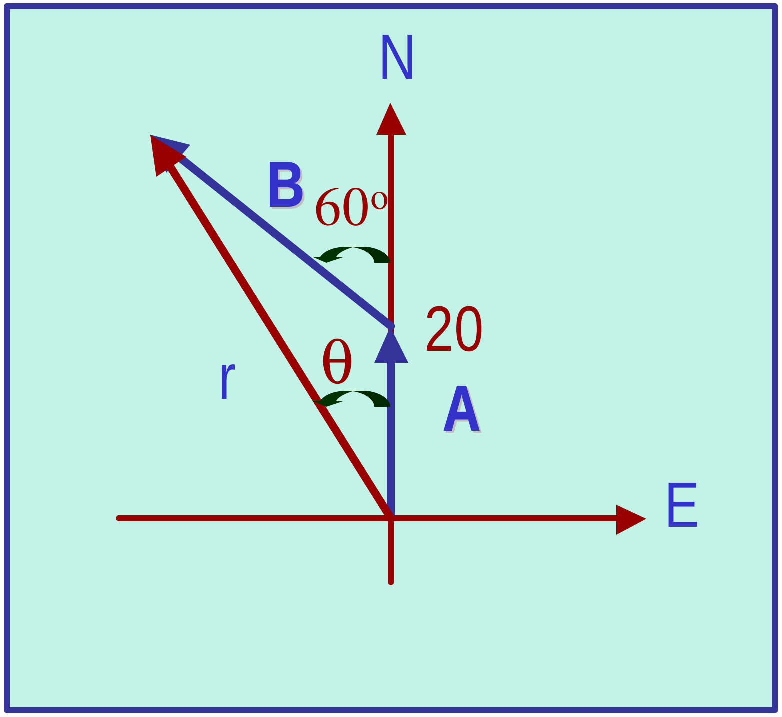
Since subtraction is the equivalent to adding a negative vector, subtraction is also commutative!!!

- Multiplication by a scalar is increasing the magnitude **A, B**=2**A**

$$\boxed{|B| = 2|A|}$$

Example 3.2

A car travels 20.0km due north followed by 35.0km in a direction 60.0º west of north. Find the magnitude and direction of resultant displacement.



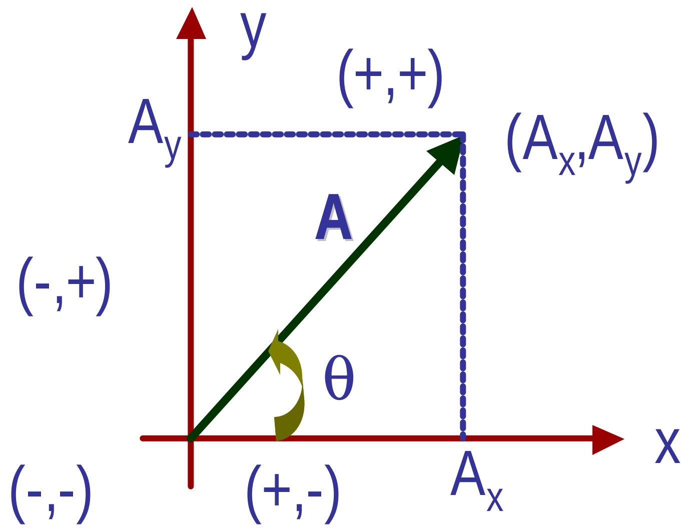
$$r = \sqrt{(A + B\cos q)^2 + (B\sin q)^2}$$
$$= \sqrt{A^2 + B^2\left(\cos^2 q + \sin^2 q\right) + 2AB\cos q}$$
$$= \sqrt{A^2 + B^2 + 2AB\cos q}$$
$$= \sqrt{(20.0)^2 + (35.0)^2 + 2\times 20.0\times 35.0\cos 60}$$
$$= \sqrt{2325} = 48.2(km)$$

$$q = \tan^{-1}\frac{\left|\vec{B}\right|\sin 60}{\left|\vec{A}\right| + \left|\vec{B}\right|\cos 60}$$
$$= \tan^{-1}\frac{35.0\sin 60}{20.0 + 35.0\cos 60}$$
$$= \tan^{-1}\frac{30.3}{37.5} = 38.9^\circ \text{ to W wrt N}$$

Find other ways to solve this problem …

Components and Unit Vectors

- Coordinate systems are useful in expressing vectors in their components



$$\left.\begin{aligned} A_x &= \left|\vec{A}\right| \cos\theta \\ A_y &= \left|\vec{A}\right| \sin\theta \end{aligned}\right\} \text{Components}$$

$$\left|\vec{A}\right| = \sqrt{A_x^2 + A_y^2}$$

$$\left|\vec{A}\right| = \sqrt{\left(\left|\vec{A}\right|\cos\theta\right)^2 + \left(\left|\vec{A}\right|\sin\theta\right)^2}$$
$$= \sqrt{\left|\vec{A}\right|^2\left(\cos^2\theta + \sin^2\theta\right)} = \left|\vec{A}\right|$$

- Unit vectors are **dimensionless** vectors whose **magnitude is exactly 1**
  - Unit vectors are usually expressed in **i,j,k** or $\vec{i}, \vec{j}, \vec{k}$
  - Vectors can be expressed using components and unit vectors

So the above vector **A** can be written as

$$\vec{A} = A_x\vec{i} + A_y\vec{j} = \left|\vec{A}\right|\cos\theta\,\vec{i} + \left|\vec{A}\right|\sin\theta\,\vec{j}$$

Examples 3.3 & 3.4

Find the resultant vector which is the sum of **A**=(2.0**i**+2.0**j**) and **B** =(2.0**i**-4.0**j**)

$$\vec{C} = \vec{A} + \vec{B} = \left(2.0\vec{i} + 2.0\vec{j}\right) + \left(2.0\vec{i} - 4.0\vec{j}\right)$$
$$= \left(2.0 + 2.0\right)\vec{i} + \left(2.0 - 4.0\right)\vec{j} = \left(4.0\vec{i} - 2.0\vec{j}\right)m$$
$$\left|\vec{C}\right| = \sqrt{(4.0)^2 + (-2.0)^2} = \sqrt{16 + 4.0} = \sqrt{20} = 4.5(m)$$
$$q = \tan^{-1}\frac{C_y}{C_x} = \tan^{-1}\frac{-2.0}{4.0} = -27^\circ$$

Find the resultant displacement of three consecutive displacements:
$\mathbf{d_1}$=(15**i**+30**j** +12**k**)cm, $\mathbf{d_2}$=(23**i**+14**j** -5.0**k**)cm, and $\mathbf{d_1}$=(-13**i**+15**j**)cm

$$\vec{D} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3$$
$$= \left(15\vec{i} + 30\vec{j} + 12\vec{k}\right) + \left(23\vec{i} - 14\vec{j} - 5.0\vec{k}\right) + \left(-13\vec{i} + 15\vec{j}\right)$$
$$= \left(15 + 23 - 13\right)\vec{i} + \left(30 - 14 + 15\right)\vec{j} + \left(12 - 5.0\right)\vec{k}$$
$$= 25\vec{i} + 31\vec{j} + 7.0\vec{k}\ (cm)$$
$$\left|\vec{D}\right| = \sqrt{(25)^2 + (31)^2 + (7.0)^2} = 40\ (cm)$$

Displacement, Velocity, and Acceleration in 2-dim

- Displacement: $$\Delta \vec{r} = \vec{r_f} - \vec{r_i}$$
- Average Velocity: $$\vec{v} \equiv \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r_f} - \vec{r_i}}{t_f - t_i}$$
- Instantaneous Velocity: $$\vec{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$
- Average Acceleration: $$\vec{a} \equiv \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v_f} - \vec{v_i}}{t_f - t_i}$$
- Instantaneous Acceleration: $$\vec{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$$

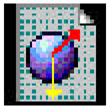
2-dim Motion Under Constant Acceleration

- Position vectors in xy plane: $\vec{r}_i = x_i\vec{i} + y_i\vec{j}$ $\quad$ $\vec{r}_f = x_f\vec{i} + y_f\vec{j}$
- Velocity vectors in xy plane: $\vec{v}_i = v_{xi}\vec{i} + v_{yi}\vec{j}$ $\quad$ $\vec{v}_f = v_{xf}\vec{i} + v_{yf}\vec{j}$

$$v_{xf} = v_{xi} + a_x t, v_{yf} = v_{yi} + a_y t$$
$$\vec{v}_f = (v_{xi} + a_x t)\vec{i} + (v_{yi} + a_y t)\vec{j} = \vec{v}_i + \vec{a}t$$

- How are the position vectors written in acceleration vectors?

$$x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2, y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$$
$$\vec{r}_f = \left(x_i + v_{xi}t + \frac{1}{2}a_x t^2\right)\vec{i} + \left(y_i + v_{yi}t + \frac{1}{2}a_y t^2\right)\vec{j}$$
$$= \vec{r}_i + \vec{v}t + \frac{1}{2}\vec{a}t^2$$



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Example 4.1

A particle starts at origin when t=0 with an initial velocity **v**=(20**i**-15**j**)m/s. The particle moves in the xy plane with $a_x=4.0m/s^2$. Determine the components of velocity vector at any time, t.

$$v_{xf} = v_{xi} + a_x t = 20 + 4.0t\,(m/s)$$

$$v_{yf} = v_{yi} + a_y t = -15\,(m/s)$$

$$\vec{v}(t) = \left\{(20 + 4.0t)\vec{i} - 15\vec{j}\right\} m/s$$

Compute the velocity and speed of the particle at t=5.0 s.

$$\vec{v} = \left\{(20 + 4.0\times 5.0)\vec{i} - 15\vec{j}\right\} m/s = \left(40\vec{i} - 15\vec{j}\right) m/s$$

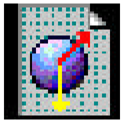
$$speed = \left|\vec{v}\right| = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(40)^2 + (-15)^2} = 43\, m/s$$

$$q = \tan^{-1}\left(\frac{-15}{40}\right) = \tan^{-1}\left(\frac{-3}{8}\right) = -21^\circ$$

Determine the *x* and *y* components of the particle at t=5.0 s.

$$x_f = v_{xi}t + \frac{1}{2}a_x t^2 = 20\times 5 + \frac{1}{2}\times 4\times 5^2 = 150\, m,\quad y_f = v_{yi}t = -15\times 5 = -75\, m$$

$$\vec{r}_f = x_f\vec{i} + y_f\vec{j} = \left(150\vec{i} - 75\vec{j}\right) m$$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Projectile Motion

- A 2-dim motion of an object under the gravitational acceleration with the assumptions
  – Free fall acceleration, -*g*, is constant over the range of the motion
  – Air resistance and other effects are negligible
- A motion under constant acceleration!!!! ➔ Superposition of two motions
  – Horizontal motion with constant velocity and
  – Vertical motion under constant acceleration

Show that a projectile motion is a parabola!!!

$$\vec{a} = a_x \vec{i} + a_y \vec{j} = -g\vec{j}$$
$$v_{xi} = v_i \cos q_i, v_{yi} = v_i \sin q_i$$
$$x_f = v_{xi} t = v_i \cos q_i t$$
$$y_f = v_{yi} t + \frac{1}{2}(-g)t^2$$
$$= v_i \sin q_i t - \frac{1}{2} g t^2$$

$$t = \frac{x_f}{v_i \cos q_i}$$
$$y_f = v_i \sin q_i \left(\frac{x_f}{v_i \cos q_i}\right) - \frac{1}{2} g \left(\frac{x_f}{v_i \cos q_i}\right)^2$$
$$= x_f \tan q_i - \left(\frac{g}{2 v_i^2 \cos^2 q_i}\right) x_f^2$$

Example 4.2

A ball is thrown with an initial velocity **v**=(20**i**+40**j**)m/s. Estimate the time of flight and the distance the ball is from the original position when landed.

Which component determines the flight time and the distance?

Flight time is determined by *y* component, because the ball stops moving when it is on the ground after the flight.

$$y_f = 40t + \frac{1}{2}(-g)t^2 = 0\,m$$

$$t(80 - gt) = 0$$

$$\therefore\ t = 0 \text{ or } t = \frac{80}{g} \approx 8\text{ sec}$$

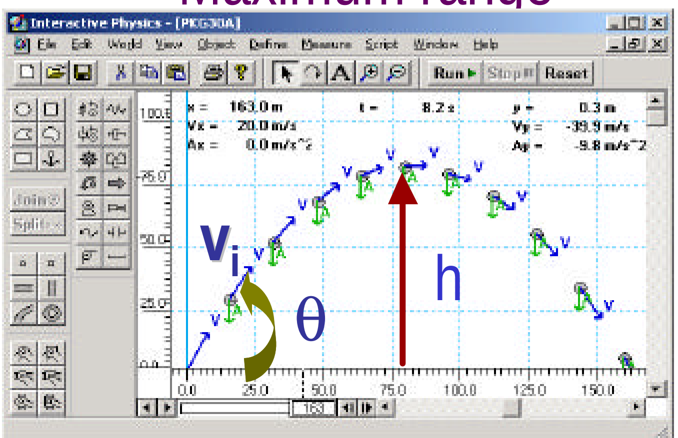
Distance is determined by *x* component in 2-dim, because the ball is at *y=0* position when it completed it's flight.

$$x_f = v_{xi}t$$

$$= 20 \times 8 = 160\ (m)$$

Horizontal Range and Max Height

- Based on what we have learned previously, one can analyze a projectile motion in more detail
  - Maximum height an object can reach
  - Maximum range



At the maximum height the object's vertical motion stops to turn around!!

$$v_{yf} = v_{yi} + a_y t = v_i \sin\theta_i - g t_A = 0$$

$$\therefore t_A = \frac{v_i \sin\theta_i}{g}$$

Since no acceleration in x, it still flies at $v_y=0$

$$R = v_{xi} 2t_A = 2 v_i \cos\theta_i \left(\frac{v_i \sin\theta_i}{g}\right) = \left(\frac{v_i^2 \sin 2\theta_i}{g}\right)$$

$$y_f = h = v_{yi} t + \frac{1}{2}(-g)t^2 = v_i \sin\theta_i \left(\frac{v_i \sin\theta_i}{g}\right) - \frac{1}{2} g \left(\frac{v_i \sin\theta_i}{g}\right)^2 = \left(\frac{v_i^2 \sin^2\theta_i}{2g}\right)$$

Maximum Range and Height

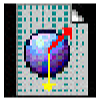
- What are the conditions that give maximum height and range in a projectile motion?

$$h=\left(\frac{v_i^2\sin^2\theta_i}{2g}\right)$$

This formula tells us that the maximum hiegth can be achieved when $\theta_i=90^o$!!!

$$R=\left(\frac{v_i^2\sin 2\theta_i}{g}\right)$$

This formula tells us that the maximum hiegth can be achieved when $2\theta_i=90^o$, i.e., $\theta_i=45^o$!!!



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Example 4.5

- A stone was thrown upward from the top of a building at an angle of 30o to horizontal with initial speed of 20.0m/s. If the height of the building is 45.0m, how long is it before the stone hits the ground?

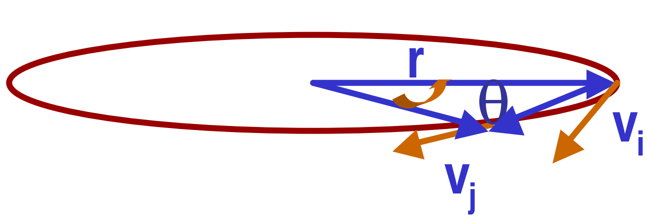
$$v_{xi} = v_i \cos q_i = 20.0 \times \cos 30^\circ = 17.3\,m/s$$
$$v_{yi} = v_i \sin q_i = 20.0 \times \sin 30^\circ = 10.0\,m/s$$
$$y_f = -45.0 = v_{yi} t - \frac{1}{2} g t^2$$
$$gt^2 - 20.0t - 90.0 = 9.80t^2 - 20.0t - 90.0 = 0$$
$$t = \frac{20.0 \pm \sqrt{(-20)^2 - 4 \times 9.80 \times (-90)}}{2 \times 9.80}$$
$$t = -2.18\,s \text{ or } t = 4.22\,s$$
$$\therefore t = 4.22\,s$$

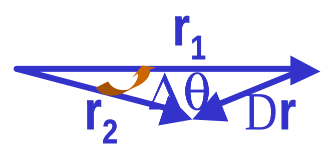
- What is the speed of the stone just before it hits the ground?

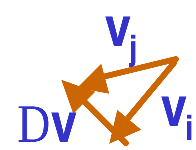
$$v_{xf} = v_{xi} = v_i \cos q_i = 20.0 \times \cos 30^\circ = 17.3\,m/s$$
$$v_{yf} = v_{yi} - gt = v_i \sin q_i - gt = 10.0 - 9.80 \times 4.22 = -31.4\,m/s$$
$$|v| = \sqrt{v_{xf}^2 + v_{yf}^2} = \sqrt{17.3^2 + (-31.4)^2} = 35.9\,m/s$$

Uniform Circular Motion

- A motion with a constant speed on a circular path.
  - The velocity of the object changes, because the direction changes
  - Therefore, there is an acceleration







Angle is Δθ

The acceleration pulls the object inward: *Centripetal Acceleration*

Average Acceleration

$$\vec{a} = \frac{\vec{v_f} - \vec{v_i}}{t_f - t_i} = \frac{\Delta \vec{v}}{\Delta t}$$

$$q = \frac{\left|\Delta \vec{v}\right|}{v} = \frac{\left|\Delta \vec{r}\right|}{r}, \quad \left|\Delta \vec{v}\right| = v\frac{\left|\Delta \vec{r}\right|}{r}, \quad \bar{a} = \frac{v}{\Delta t}\frac{\left|\Delta \vec{r}\right|}{r}$$

Instantaneous Acceleration

$$a_r = \lim_{\Delta t \to 0} \bar{a} = \lim_{\Delta t \to 0} \frac{\left|\Delta \vec{r}\right|}{\Delta t}\frac{v}{r} = v \times \frac{v}{r} = \frac{v^2}{r}$$

Is this correct in dimension?

What story is this expression telling you?