1443-501 Spring 2002
Lecture #13

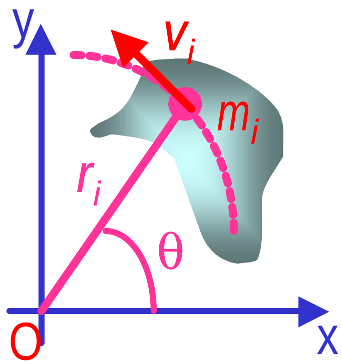
Dr. Jaehoon Yu

1. Rotational Energy
2. Computation of Moments of Inertia
3. Parallel-axis Theorem
4. Torque & Angular Acceleration
5. Work, Power, & Energy of Rotational Motions

Remember the mid-term exam on Wednesday, Mar. 13. Will cover Chapters 1-10.

Today's Homework Assignment is the Homework #5!!!

Rotational Energy



What do you think the kinetic energy of a rigid object that is undergoing a circular motion is?

Kinetic energy of a masslet, m_i, moving at a tangential speed, v_i, is

$$K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 w^2$$

Since a rigid body is a collection of masslets, the total kinetic energy of the rigid object is

$$K_R = \sum_i K_i = \frac{1}{2} \sum_i m_i r_i^2 w^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right) w^2$$

By defining a new quantity called, Moment of Inertia, *I*, as

$$I = \sum_i m_i r_i^2$$

The above expression is simplified as

$$K_R = \frac{1}{2} I w^2$$

What are the dimension and unit of Moment of Inertia? $kg \cdot m^2$ $\left[ML^2\right]$

What do you think the moment of inertia is?

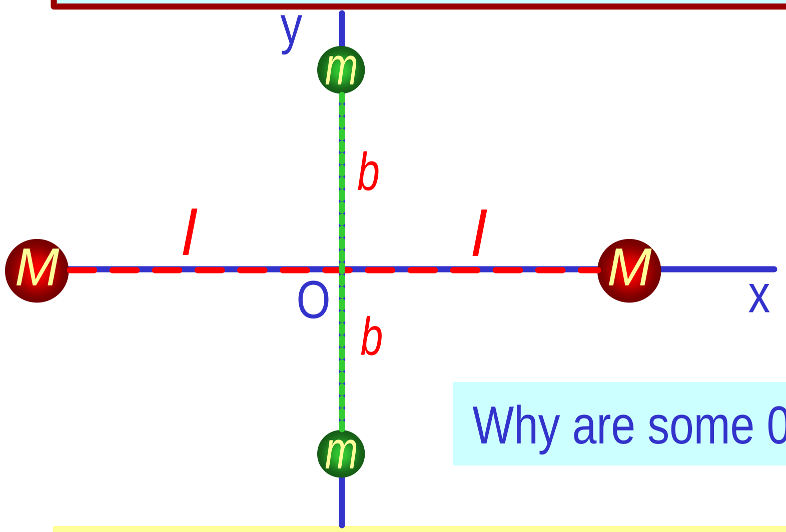
Measure of resistance of an object to changes in its rotational motion.

What similarity do you see between rotational and linear kinetic energies?

Mass and speed in linear kinetic energy are replaced by moment of inertia and angular speed.

Example 10.4

In a system consists of four small spheres as shown in the figure, assuming the radii are negligible and the rods connecting the particles are massless, compute the moment of inertia and the rotational kinetic energy when the system rotates about the y-axis at ω.



Since the rotation is about y axis, the moment of inertia about y axis, I_y, is

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + m \cdot 0^2 + m \cdot 0^2 = 2Ml^2$$

Why are some 0s?

This is because the rotation is done about y axis, and the radii of the spheres are negligible.

Thus, the rotational kinetic energy is

$$K_R = \frac{1}{2} I w^2 = \frac{1}{2}\left(2Ml^2\right)w^2 = Ml^2 w^2$$

Find the moment of inertia and rotational kinetic energy when the system rotates on the x-y plane about the z-axis that goes through the origin O.

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + mb^2 + mb^2 = 2\left(Ml^2 + mb^2\right)$$

$$K_R = \frac{1}{2} I w^2 = \frac{1}{2}\left(2Ml^2 + 2mb^2\right)w^2 = \left(Ml^2 + mb^2\right)w^2$$

Calculation of Moments of Inertia

Moments of inertia for large objects can be computed, if we assume the object consists of small volume elements with mass, Δm_i.

The moment of inertia for the large rigid object is $I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$

It is sometimes easier to compute moments of inertia in terms of volume of the elements rather than their mass

How can we do this?

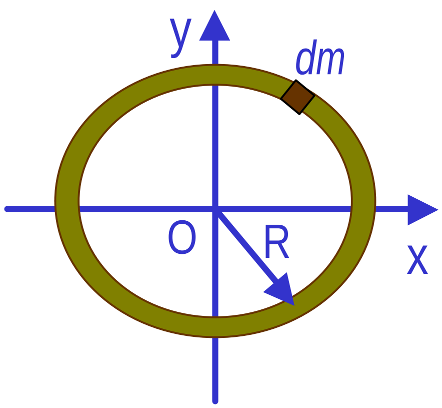
Using the volume density, ρ, replace d*m* in the above equation with dV.

$$\rho = \frac{dm}{dV}; \quad dm = \rho dV$$

The moments of inertia becomes

$$I = \int \rho r^2 dV$$

Example 10.5: Find the moment of inertia of a uniform hoop of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center.



The moment of inertia is

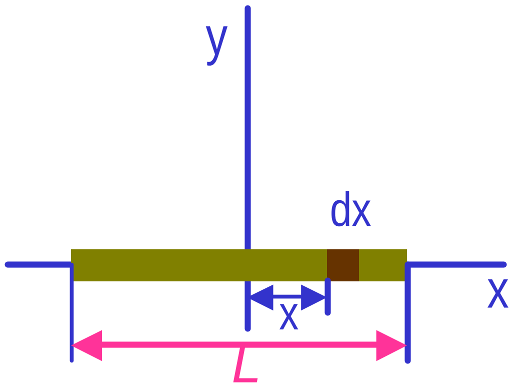
$$I = \int r^2 dm = R^2 \int dm = MR^2$$

What do you notice from this result?

The moment of inertia for this object is the same as that of a point of mass M at the distance R.

Example 10.6

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod and passing through its center of mass.



The line density of the rod is $l = \frac{M}{L}$

so the masslet is $dm = l dx = \frac{M}{L} dx$

The moment of inertia is

$$I = \int r^2 dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

What is the moment of inertia when the rotational axis is at one end of the rod.

$$I = \int r^2 dm = \int_0^L \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_0^L$$

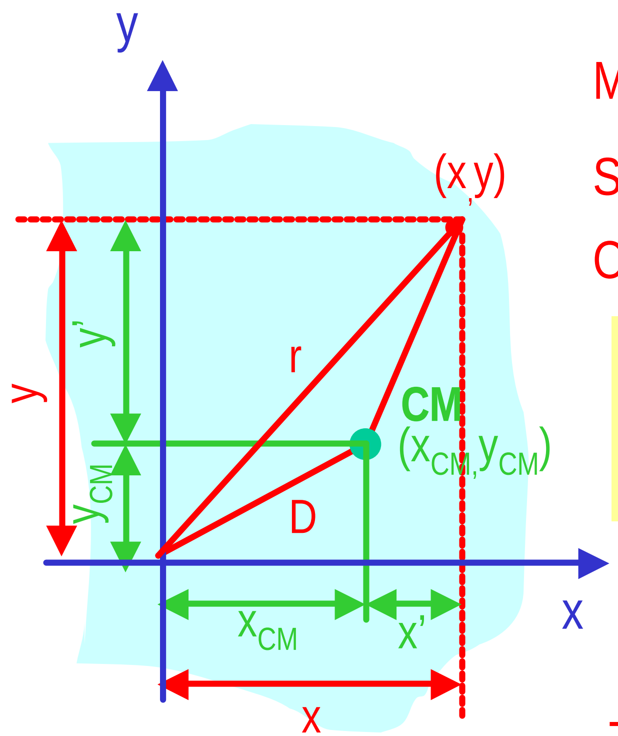
$$= \frac{M}{3L}\left[(L)^3 - 0\right] = \frac{M}{3L}\left(L^3\right) = \frac{ML^2}{3}$$

Will this be the same as the above. Why or why not?

Since the moment of inertia is resistance to motion, it makes perfect sense for it to be harder to move when it is rotating about the axis at one end.

Parallel Axis Theorem

Moments of inertia for highly symmetric object is relatively easy if the rotational axis is the same as the axis of symmetry. However if the axis of rotation does not coincide with axis of symmetry, the calculation can still be done in simple manner using **parallel-axis theorem**. $I = I_{CM} + MD^2$



Moment of inertia is defined $I = \int r^2 dm = \int \sqrt{(x^2+y^2)}\,dm \quad (1)$

Since x and y are $x = x_{CM} + x'; \quad y = y_{CM} + y'$

One can substitute x and y in Eq. 1 to obtain

$$I = \int\left[(x_{CM}+x')^2 + (y_{CM}+y')^2\right]dm$$
$$= \left(x_{CM}^2 + y_{CM}^2\right)\int dm + 2x_{CM}\int x'\,dm + 2y_{CM}\int y'\,dm + \int\left(x'^2 + y'^2\right)dm$$

Since the x' and y' are the distance from CM, by definition

$$\int x'\,dm = 0$$
$$\int y'\,dm = 0$$

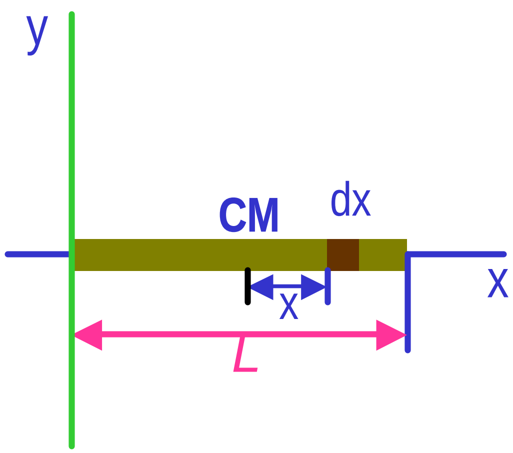
Therefore, the parallel-axis theorem $I = I_{CM} + MD^2$

What does this theorem tell you?

Moment of inertia of any object about any arbitrary axis are the same as the sum of moment of inertia for a rotation about the CM and that of the CM about the rotation axis.

Example 10.8

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis that goes through one end of the rod, using parallel-axis theorem.



The line density of the rod is

$$\lambda = \frac{M}{L}$$

so the masslet is

$$dm = \lambda dx = \frac{M}{L} dx$$

The moment of inertia about the CM

$$I_{CM} = \int r^2 dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

Using the parallel axis theorem

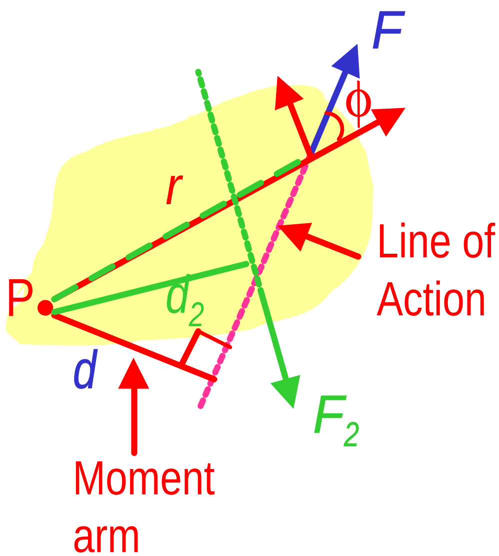
$$I = I_{CM} + D^2 M = \frac{ML^2}{12} + \left(\frac{L}{2}\right)^2 M = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}$$

The result is the same as using the definition of moment of inertia.

Parallel-axis theorem is useful to compute moment of inertia of a rotation of a rigid object with complicated shape about an arbitrary axis

Torque

Torque is the tendency of a force to rotate an object about some axis. Torque, t, is a vector quantity.



Consider an object pivoting about the point P by the force *F* being exerted at a distance r.

The line that extends out of the tail of the force vector is called the line of action.

The perpendicular distance from the pivoting point P to the line of action is called Moment arm.

Magnitude of torque is defined as the product of the force exerted on the object to rotate it and the moment arm.

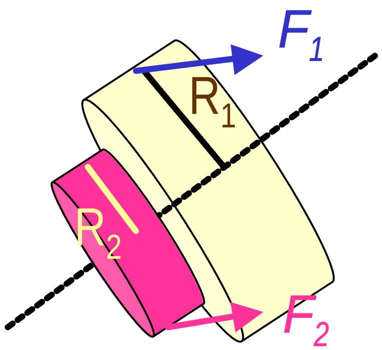
$$t \equiv rF\sin f = Fd$$

When there are more than one force being exerted on certain points of the object, one can sum up the torque generated by each force vectorially. The convention for sign of the torque is positive if rotation is in counter-clockwise and negative if clockwise.

$$\sum t = t_1 + t_2 = Fd - F_2 d_2$$

Example 10.9

A one piece cylinder is shaped as in the figure with core section protruding from the larger drum. The cylinder is free to rotate around the central axis shown in the picture. A rope wrapped around the drum whose radius is R_1 exerts force F_1 to the right on the cylinder, and another force exerts F_2 on the core whose radius is R_2 downward on the cylinder. A) What is the net torque acting on the cylinder about the rotation axis?



The torque due to F_1 $\quad t_1 = -R_1F_1$ $\quad$ and due to F_2 $\quad t_2 = R_2F_2$

So the total torque acting on the system by the forces is

$$\sum t = t_1 + t_2 = -R_1F_1 + -R_2F_2$$

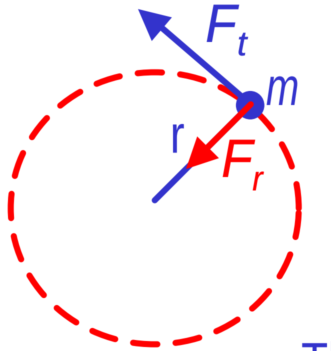
Suppose F_1=5.0 N, R_1=1.0 m, F_2= 15.0 N, and R_2=0.50 m. What is the net torque about the rotation axis and which way does the cylinder rotate from the rest?

Using the above result

$$\sum t = -R_1F_1 + R_2F_2$$
$$= -5.0\times1.0 + 15.0\times0.50 = 2.5N \bullet m$$

The cylinder rotates in counter-clockwise.

Torque & Angular Acceleration



Let's consider a point object with mass *m* rotating in a circle.

What forces do you see in this motion?

The tangential force F_t and radial force F_r

The tangential force F_t is $F_t = ma_t = mra$

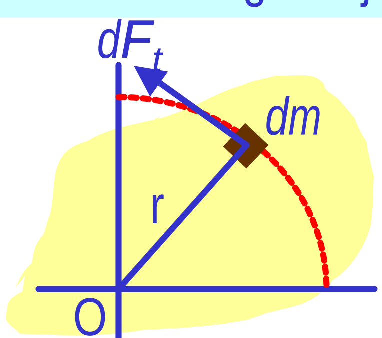
The torque due to tangential force F_t is $t = F_t r = ma_t r = mr^2 a$

What do you see from the above relationship? $t = Ia$

What does this mean? Torque acting on a particle is proportional to the angular acceleration.

What law do you see from this relationship? Analogs to Newton's 2nd law of motion in rotation.

How about a rigid object?



The external tangential force dF_t is $dF_t = dma_t = dmra$

The torque due to tangential force F_t is $dt = dF_t r = \left(r^2 dm\right)a$

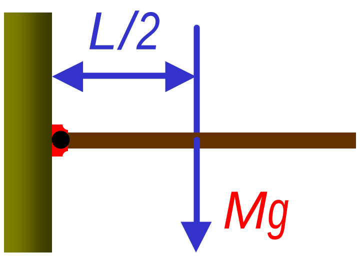
The total torque is $\sum t = a \int r^2 dm = Ia$

What is the contribution due to radial force and why?

Contribution from radial force is 0, because its line of action passes through the pivoting point, making the moment arm 0.

Example 10.10

A uniform rod of length *L* and mass *M* is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane. The rod is released from rest in the horizontal position what is the initial angular acceleration of the rod and the initial linear acceleration of its right end?



The only force generating torque is the gravitational force *Mg*

$$\tau = Fd = F\frac{L}{2} = Mg\frac{L}{2} = I\alpha$$

Since the moment of inertia of the rod when it rotates about one end

$$I = \int_0^L r^2 dm = \int_0^L x^2 \lambda dx = \left(\frac{M}{L}\right)\left[\frac{x^3}{3}\right]_0^L = \frac{ML^2}{3}$$

We obtain

$$\alpha = \frac{MgL}{2I} = \frac{MgL}{\frac{2ML^2}{3}} = \frac{3g}{2L}$$

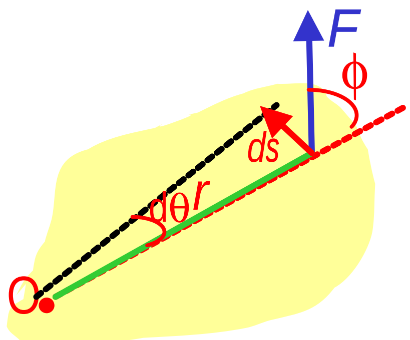
Using the relationship between tangential and angular acceleration

$$a_t = L\alpha = \frac{3g}{2}$$

What does this mean?

The tip of the rod falls faster than an object undergoing a free fall.

Work, Power, and Energy in Rotation



Let's consider a motion of a rigid body with a single external force *F* exerted on the point P, moving the object by *ds*.

The work done by the force *F* as the object rotates through infinitesimal distance ds=rdθ in a time dt is

$$dW = \vec{F} \cdot d\vec{s} = (F \sin f) r dq$$

What is *F*sinφ? The tangential component of force *F*.

What is the work done by radial component *F*cosφ? Zero, because it is perpendicular to the displacement.

Since the magnitude of torque is r*F*sinφ,

$$dW = t dq$$

The rate of work, or power becomes

$$P = \frac{dW}{dt} = \frac{t dq}{dt} = t w$$

How was the power defined in linear motion?

The rotational work done by an external force equals the change in rotational energy.

$$\sum t = I a = I\left(\frac{dw}{dt}\right) = I\left(\frac{dw}{dq}\right)\left(\frac{dq}{dt}\right)$$

The work put in by the external force then

$$dW = \sum t\, dq = I w dw$$

$$\sum W = \int_{q_i}^{q_i} \sum t\, dq = \int_{w_i}^{w_f} I w dw = \frac{1}{2} I w_i^2 - \frac{1}{2} I w_f^2$$

Similarity Between Linear and Rotational Motions

All physical quantities in linear and rotational motions show striking similarity.

Similar Quantity	Linear	Rotational
Mass	Mass M	Moment of Inertia $I = \int r^2 dm$
Length of motion	Distance L	Angle q (Radian)
Speed	$v = \frac{dr}{dt}$	$w = \frac{dq}{dt}$
Acceleration	$a = \frac{dv}{dt}$	$a = \frac{dw}{dt}$
Force	Force $F = ma$	Torque $t = Ia$
Work	Work $W = \int_{x_i}^{x_f} F dx$	Work $W = \int_{q_i}^{q_f} t\, dq$
Power	$P = \vec{F} \cdot \vec{v}$	$P = tw$
Momentum	$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{w}$
Kinetic Energy	Kinetic $K = \frac{1}{2}mv^2$	Rotational $K_R = \frac{1}{2}Iw^2$