1443-501 Spring 2002
Lecture #14

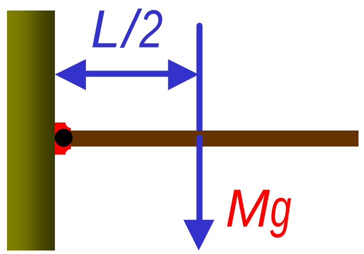
Dr. Jaehoon Yu

1. Work, Power, & Energy of Rotational Motions
2. Review examples of Chapters 1 - 10

Mid-term exam 5:30-6:50pm, this Wednesday, Mar. 13, in the classroom.
Bring your own blue books for additional answer sheets.

Example 10.10

A uniform rod of length *L* and mass *M* is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane. The rod is released from rest in the horizontal position what is the initial angular acceleration of the rod and the initial linear acceleration of its right end?



The only force generating torque is the gravitational force *Mg*

$$t = Fd = F\frac{L}{2} = Mg\frac{L}{2} = Ia$$

Since the moment of inertia of the rod when it rotates about one end is

$$I = \int_0^L r^2 dm = \int_0^L x^2 l dx = \left(\frac{M}{L}\right)\left[\frac{x^3}{3}\right]_0^L = \frac{ML^2}{3}$$

We obtain

$$a = \frac{MgL}{2I} = \frac{MgL}{\frac{2ML^2}{3}} = \frac{3g}{2L}$$

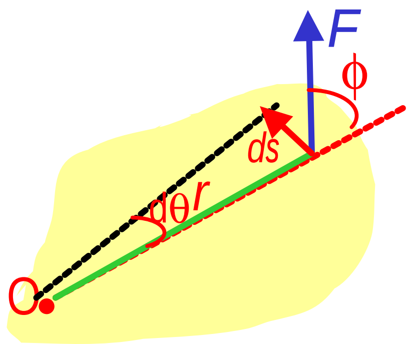
Using the relationship between tangential and angular acceleration

$$a_t = La = \frac{3g}{2}$$

What does this mean?

The tip of the rod falls faster than an object undergoing a free fall.

Work, Power, and Energy in Rotation



Let's consider a motion of a rigid body with a single external force *F* exerted on the point P, moving the object by *ds*.

The work done by the force *F* as the object rotates through infinitesimal distance ds=rdθ in a time dt is

$$dW = \vec{F}\cdot d\vec{s} = (F\sin f)rdq$$

What is *F*sinϕ? — The tangential component of force *F*.

What is the work done by radial component *F*cosϕ? — Zero, because it is perpendicular to the displacement.

Since the magnitude of torque is r*F*sinϕ,

$$dW = t dq$$

The rate of work, or power becomes

$$P = \frac{dW}{dt} = \frac{t dq}{dt} = t w$$

How was the power defined in linear motion?

$$\sum t = Ia = I\left(\frac{dw}{dt}\right) = I\left(\frac{dw}{dq}\right)\left(\frac{dq}{dt}\right)$$

The rotational work done by an external force equals the change in rotational energy.

The work put in by the external force then

$$dW = \sum t\, dq = Iwdw$$

$$\sum W = \int_{q_i}^{q_i} \sum t\, dq = \int_{w_i}^{w_f} Iwdw = \frac{1}{2}Iw_i^2 - \frac{1}{2}Iw_f^2$$

Similarity Between Linear and Rotational Motions

All physical quantities in linear and rotational motions show striking similarity.

Similar Quantity	Linear	Rotational
Mass	Mass M	Moment of Inertia $I = \int r^2 dm$
Length of motion	Distance L	Angle q (Radian)
Speed	$v = \frac{dr}{dt}$	$w = \frac{dq}{dt}$
Acceleration	$a = \frac{dv}{dt}$	$a = \frac{dw}{dt}$
Force	Force $F = ma$	Torque $t = Ia$
Work	Work $W = \int_{x_i}^{x_f} F dx$	Work $W = \int_{q_i}^{q_f} t\, dq$
Power	$P = \vec{F} \cdot \vec{v}$	$P = tw$
Momentum	$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{w}$
Kinetic Energy	Kinetic $K = \frac{1}{2}mv^2$	Rotational $K_R = \frac{1}{2}Iw^2$

2-dim Motion Under Constant Acceleration

- Position vectors in xy plane: $\vec{r}_i = x_i\vec{i} + y_i\vec{j}$ $\vec{r}_f = x_f\vec{i} + y_f\vec{j}$
- Velocity vectors in xy plane: $\vec{v}_i = v_{xi}\vec{i} + v_{yi}\vec{j}$ $\vec{v}_f = v_{xf}\vec{i} + v_{yf}\vec{j}$

$$v_{xf} = v_{xi} + a_x t, v_{yf} = v_{yi} + a_y t$$
$$\vec{v}_f = (v_{xi} + a_x t)\vec{i} + (v_{yi} + a_y t)\vec{j} = \vec{v}_i + \vec{a}t$$

- How are the position vectors written in acceleration vectors?

$$x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2, y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$$
$$\vec{r}_f = \left(x_i + v_{xi}t + \frac{1}{2}a_x t^2\right)\vec{i} + \left(y_i + v_{yi}t + \frac{1}{2}a_y t^2\right)\vec{j}$$
$$= \vec{r}_i + \vec{v}t + \frac{1}{2}\vec{a}t^2$$

Example 2.12

A stone was thrown straight upward at t=0 with +20.0m/s initial velocity on the roof of a 50.0m high building,

1. Find the time the stone reaches at maximum height (*v=0*)
2. Find the maximum height
3. Find the time the stone reaches its original height
4. Find the velocity of the stone when it reaches its original height
5. Find the velocity and position of the stone at t=5.00s

$g=-9.80m/s^2$

1

$$v_f = v_{yi} + a_y t = +20.0 - 9.80t = 0.00$$

$$t = \frac{20.0}{9.80} = 2.04s$$

2

$$y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$$

$$= 50.0 + 20\times 2.04 + \frac{1}{2}\times(-9.80)\times(2.04)^2$$

$$= 50.0 + 20.4 = 70.4(m)$$

3

$$t = 2.04\times 2 = 4.08s$$

4

$$v_{yf} = v_{yi} + a_y t = 20.0 + (-9.80)\times 4.08 = -20.0(m/s)$$

5-Velocity

$$v_{yf} = v_{yi} + a_y t$$

$$= 20.0 + (-9.80)\times 5.00$$

$$= -29.0(m/s)$$

5-Position

$$y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$$

$$= 50.0 + 20.0\times 5.00 + \frac{1}{2}\times(-9.80)\times(5.00)^2 = +27.5(m)$$

Example 4.5

- A stone was thrown upward from the top of a building at an angle of 30° to horizontal with initial speed of 20.0m/s. If the height of the building is 45.0m, how long is it before the stone hits the ground?

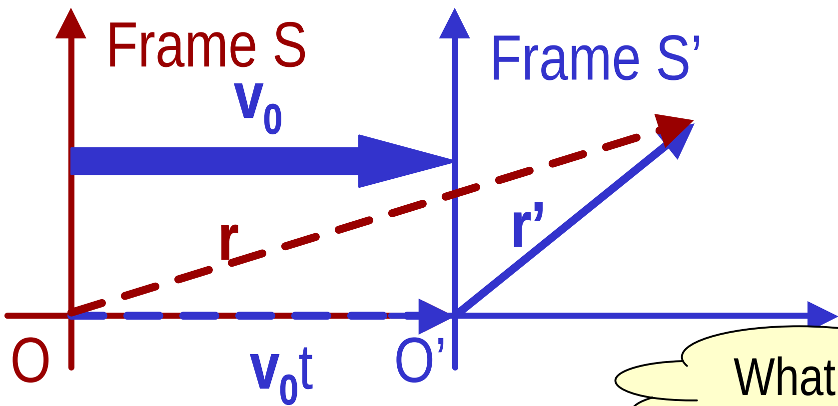
$$v_{xi} = v_i \cos\theta_i = 20.0 \times \cos 30^\circ = 17.3\,m/s$$
$$v_{yi} = v_i \sin\theta_i = 20.0 \times \sin 30^\circ = 10.0\,m/s$$
$$y_f = -45.0 = v_{yi}t - \frac{1}{2}gt^2$$
$$gt^2 - 20.0t - 90.0 = 9.80t^2 - 20.0t - 90.0 = 0$$
$$t = \frac{20.0 \pm \sqrt{(-20)^2 - 4 \times 9.80 \times (-90)}}{2 \times 9.80}$$
$$t = -2.18\,s \text{ or } t = 4.22\,s$$
$$\therefore t = 4.22\,s$$

- What is the speed of the stone just before it hits the ground?

$$v_{xf} = v_{xi} = v_i \cos\theta_i = 20.0 \times \cos 30^\circ = 17.3\,m/s$$
$$v_{yf} = v_{yi} - gt = v_i \sin\theta_i - gt = 10.0 - 9.80 \times 4.22 = -31.4\,m/s$$
$$|v| = \sqrt{v_{xf}^2 + v_{yf}^2} = \sqrt{17.3^2 + (-31.4)^2} = 35.9\,m/s$$

Relative Velocity and Acceleration

The velocity and acceleration in two different frames of references can be denoted:



Galilean transformation equation

$$\vec{r}\,' = \vec{r} - \vec{v}_0 t$$

$$\frac{d\vec{r}\,'}{dt} = \frac{d\vec{r}}{dt} - \vec{v}_0$$

$$\vec{v}\,' = \vec{v} - \vec{v}_0$$

What does this tell you?

$$\vec{r}\,' = \vec{r} - \vec{v}_0 t$$

$$\frac{d\vec{v}\,'}{dt} = \frac{d\vec{v}}{dt} - \frac{d\vec{v}_0}{dt}$$

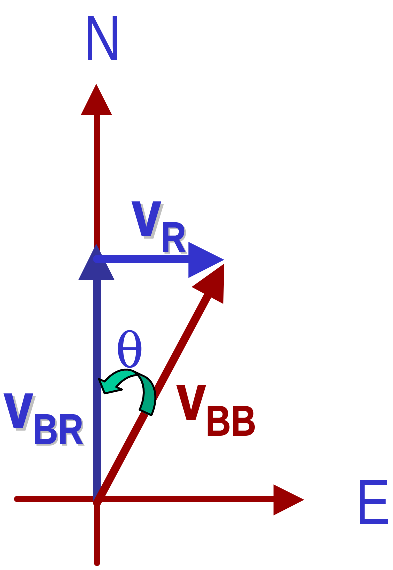
$$\vec{a}\,' = \vec{a}, \text{ when } \vec{v}_0 \text{ is constant}$$

The accelerations measured in two frames are the same when the frames move at a constant velocity with respect to each other!!!

The earth's gravitational acceleration is the same in a frame moving at a constant velocity wrt the earth.

Example 4.9

A boat heading due north with a speed 10.0km/h is crossing the river whose stream has a uniform speed of 5.00km/h due east. Determine the velocity of the boat seen by the observer on the bank.



$$\overrightarrow{v_{BB}} = \overrightarrow{v_{BR}} + \overrightarrow{v_R}$$

$$\left|\overrightarrow{v_{BB}}\right| = \sqrt{\left|\overrightarrow{v_{BR}}\right|^2 + \left|\overrightarrow{v_R}\right|^2} = \sqrt{(10.0)^2 + (5.00)^2} = 11.2\,km/h$$

$$\because \overrightarrow{v_{BR}} = 10.0\,\hat{j} \text{ and } \overrightarrow{v_R} = 5.00\,\hat{i}$$

$$\overrightarrow{v_{BB}} = 5.00\,\hat{i} + 10.0\,\hat{j}$$

$$\theta = \tan^{-1}\left(\frac{v_{BBx}}{v_{BBy}}\right) = \tan^{-1}\left(\frac{5.00}{10.0}\right) = 26.6^\circ$$

How long would it take for the boat to cross the river if the width is 3.0km?

$$v_{BB}\cos\theta \bullet t = 3.0km$$

$$t = \frac{3.0}{v_{BB}\cos\theta} = \frac{3.0}{11.2\times\cos(26.6^\circ)} = 0.30hrs = 18\min$$

Newton's Laws

1st Law:
Law of Inertia

In the absence of external forces, an object at rest remains at rest and an object in motion continues in motion with a constant velocity.

2nd Law:
Law of Forces

$$\boxed{\sum_i \vec{F}_i = m\vec{a}}$$

The acceleration of an object is directly proportional to the net force exerted on it and inversely proportional to the object's mass.

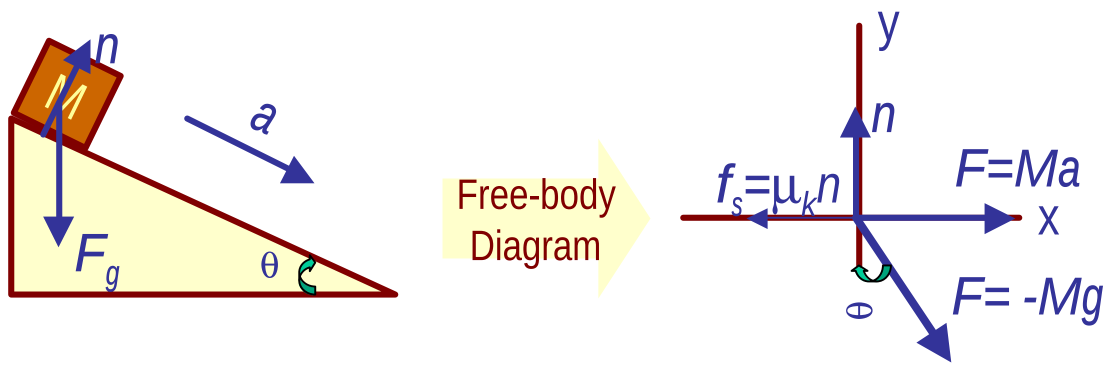
3rd Law:
Law of Action and Reaction

$$\boxed{\overrightarrow{F_{21}} = -\overrightarrow{F_{12}}}$$

If two objects interact, the force, F_{12}, exerted on object 1 by object 2 is equal magnitude to and opposite direction to the force, F_{21}, exerted on object 1 by object 2.

Example 5.12

Suppose a block is placed on a rough surface inclined relative to the horizontal. The inclination angle is increased till the block starts to move. Show that by measuring this critical angle, θ_c, one can determine coefficient of static friction, μ_s.



$$\vec{F} = \vec{F_g} + \vec{n} + f_s$$

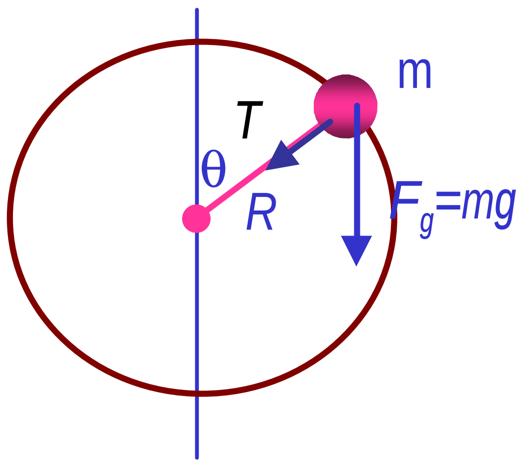
$$F_y = Ma_y = n + F_{gy} = 0; \quad n = -F_{gy} = Mg\cos\theta_c$$

$$F_x = F_{gx} - f_s = 0; \quad f_s = \mu_s n = Mg\sin\theta_c$$

$$\mu_s = \frac{Mg\sin\theta_c}{n} = \frac{Mg\sin\theta_c}{Mg\cos\theta_c} = \tan\theta_c$$

Example 6.8

A ball of mass m is attached to the end of a cord of length R. The ball is moving in a vertical circle. Determine the tension of the cord at any instant when the speed of the ball is *v* and the cord makes an angle θ with vertical.



What are the forces involved in this motion?

The gravitational force F_g and the radial force, T, providing tension.

$$\sum F_t = ma_t = mg \sin q$$

$$a_t = g \sin q$$

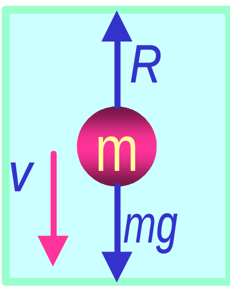
$$\sum F_r = T - mg \cos q = ma_r = m\frac{v^2}{R}$$

$$T = m\left(\frac{v^2}{R} + g \cos q\right)$$

At what angles the tension becomes maximum and minimum. What are the tension?

Example 6.11

A small ball of mass 2.00g is released from rest in a large vessel filled with oil, where it experiences a resistive force proportional to its speed. The ball reaches a terminal speed of 5.00 cm/s. Determine the time constant τ and the time it takes the ball to reach 90% of its terminal speed.



Determine the time constant τ.

$$v_t = \frac{mg}{b}$$

$$\therefore b = \frac{mg}{v_t} = \frac{2.00\times10^{-3}kg \cdot 9.80m/s^2}{5.00\times10^{-2}m/s} = 0.392kg/s$$

$$\tau = \frac{m}{b} = \frac{2.00\times10^{-3}kg}{0.392kg/s} = 5.10\times10^{-3}s$$

Determine the time it takes the ball to reach 90% of its terminal speed.

$$v = \frac{mg}{b}\left(1-e^{-t/\tau}\right) = v_t\left(1-e^{-t/\tau}\right)$$

$$0.9v_t = v_t\left(1-e^{-t/\tau}\right)$$

$$\left(1-e^{-t/\tau}\right) = 0.9;\ e^{-t/\tau} = 0.1$$

$$t = -\tau \cdot \ln 0.1 = 2.30\tau = 2.30 \cdot 5.10\times10^{-3} = 11.7(ms)$$

Work and Kinetic Energy

Work in physics is done only when a sum of forces exerted on an object made a motion to the object.

What does this mean?

However much tired your arms feel, if you were just holding an object without moving it you have not done any physical work.

Mathematically, work is written in scalar product of force vector and the displacement vector

$$W = \sum \vec{F}_i \cdot \vec{d} = Fd\cos\theta$$

Kinetic Energy is the energy associated with motion and capacity to perform work. Work requires change of energy after the completion ← Work-Kinetic energy theorem

$$K = \frac{1}{2} mv^2$$

N.m=Joule

$$\sum W = K_f - K_i = \Delta K$$

Power is the rate of which work is performed.

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d}{dt}\left(\vec{s}\right) = \vec{F} \cdot \vec{v}$$

Units of these quantities????

Nm/s=Joule/s=Watt

Example 7.14

A compact car has a mass of 800kg, and its efficiency is rated at 18%. Find the amount of gasoline used to accelerate the car from rest to 27m/s (~60mi/h). Use the fact that the energy equivalent of 1gal of gasoline is $1.3\text{x}10^8$J.

First let's compute what the kinetic energy needed to accelerate the car from rest to a speed *v*.

$$K_f = \frac{1}{2}mv^2 = \frac{1}{2}\times 800\times(27)^2 = 2.9\times 10^5\,J$$

Since the engine is only 18% efficient we must divide the necessary kinetic energy with this efficiency in order to figure out what the total energy needed is.

$$W_E = \frac{K_f}{e} = \frac{1}{2e}mv^2 = \frac{2.9\times 10^5\,J}{0.18} = 16\times 10^5\,J$$

Then using the fact that 1gal of gasoline can putout $1.3\text{x}10^8$J, we can compute the total volume of gasoline needed to accelerate the car to 60 mi/h.

$$V_{gas} = \frac{W_E}{1.3\times 10^8\,J/gal} = \frac{16\times 10^5\,J}{1.3\times 10^8\,J/gal} = 0.012\,gal$$

Potential Energy

Energy associated with a system of objects → Stored energy which has Potential or possibility to work or to convert to kinetic energy

What does this mean?

In order to describe potential energy, U, a system must be defined.

The concept of potential energy can only be used under the special class of forces called, conservative forces which results in principle of conservation of mechanical energy.

What other forms of energies in the universe?

Mechanical Energy

Chemical Energy

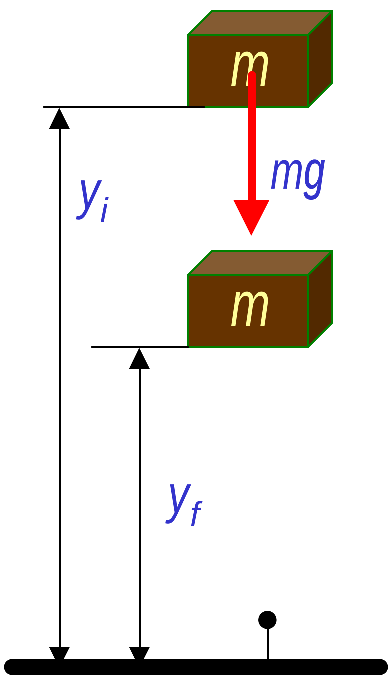
Biological Energy

Electromagnetic Energy

Nuclear Energy

Gravitational Potential

Potential energy given to an object by gravitational field in the system of Earth due to its height from the surface



When an object is falling, gravitational force, Mg, performs work on the object, increasing its kinetic energy. The potential energy of an object at a height y which is the potential to work is expressed as

$$U_g = \vec{F}_g \cdot \vec{y} = mg\left(-\vec{j}\right)\cdot y\left(-\vec{j}\right) = mgy$$

$$U_g \equiv mgy$$

Work performed on the object by the gravitational force as the brick goes from y_i to y_f is:

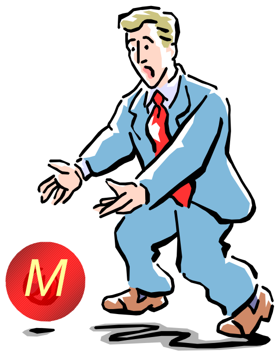
$$W_g = U_i - U_f$$
$$= mgy_i - mgy_f = -\Delta U_g$$

What does this mean?

Work by the gravitational force as the brick goes from y_i to y_f is negative of the change in the system's potential energy

Example 8.1

A bowler drops bowling ball of mass 7kg on his toe. Choosing floor level as y=0, estimate the total work done on the ball by the gravitational force as the ball falls.



Let's assume the top of the toe is 0.03m from the floor and the hand was 0.5m above the floor.

$$U_i = mgy_i = 7 \times 9.8 \times 0.5 = 34.3 J$$
$$U_f = mgy_f = 7 \times 9.8 \times 0.03 = 2.06 J$$
$$\Delta U = -(U_f - U_i) = 32.24 J \cong 30 J$$

b) Perform the same calculation using the top of the bowler's head as the origin.

What has to change?

First we must re-compute the positions of ball at the hand and of the toe.

Assuming the bowler's height is 1.8m, the ball's original position is –1.3m, and the toe is at –1.77m.

$$U_i = mgy_i = 7 \times 9.8 \times (-1.3) = -89.2 J$$
$$U_f = mgy_f = 7 \times 9.8 \times (-1.77) = -121.4 J$$
$$\Delta U = -(U_f - U_i) = 32.2 J \cong 30 J$$

Elastic Potential Energy

Potential energy given to an object by a spring or an object with elasticity in the system consists of the object and the spring without friction.

The force spring exerts on an object when it is distorted from its equilibrium by a distance x is

$$F_s = -kx$$

The work performed on the object by the spring is

$$W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$$

The potential energy of this system is

$$U_s \equiv \frac{1}{2}kx^2$$

What do you see from the above equations?

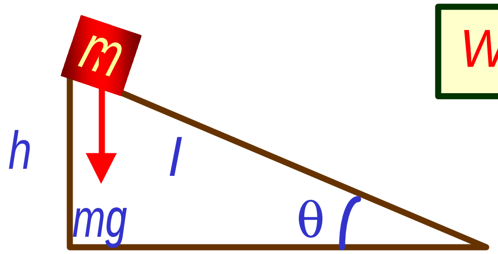
The work done on the object by the spring depends only on the initial and final position of the distorted spring.

Where else did you see this trend?

The gravitational potential energy, U_g

Conservative and Non-conservative Forces

The work done on an object by the gravitational force does not depend on the object's path.



When directly falls, the work done on the object is $W_g = mgh$

When sliding down the hill of length l, the work is

$$W_g = F_{g-incline} \times l = mg \sin\theta \times l$$
$$= mg(l\sin\theta) = mgh$$

How about if we lengthen the incline by a factor of 2, keeping the height the same??

Still the same amount of work☺ $W_g = mgh$

So the work done by the gravitational force on an object is independent on the path of the object's movements. It only depends on the difference of the object's initial and final position in the direction of the force.

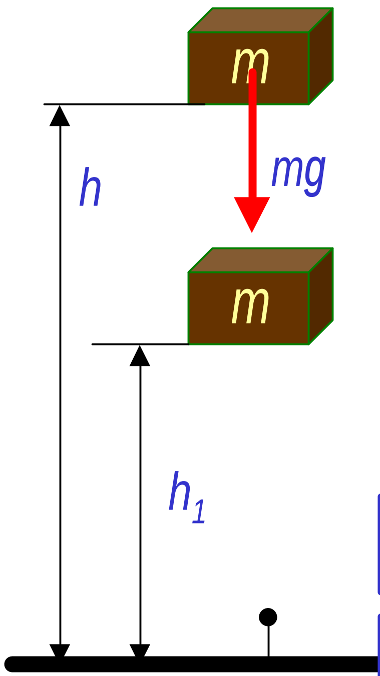
The forces like gravitational or elastic forces are called conservative forces

1. *If the work performed by the force does not depend on the path*
2. *If the work performed on a closed path is 0.*

Conservation of Mechanical Energy

Total mechanical energy is the sum of kinetic and potential energies

$$E \equiv K + U$$



Let's consider a brick of mass *m* at a height *h* from the ground

What is its potential energy?

$$U_g = mgh$$

What happens to the energy as the brick falls to the ground?

$$\Delta U = U_f - U_i = -\int_{x_i}^{x_f} F_x dx$$

The brick gains speed

By how much?

$$v = gt$$

So what?

The brick's kinetic energy increased

$$K = \frac{1}{2} mv^2 = \frac{1}{2} mg^2 t^2$$

And?

The lost potential energy is converted to kinetic energy

What does this mean?

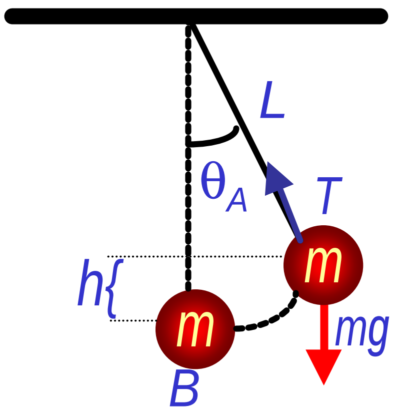
The total mechanical energy of a system remains constant in any isolated system of objects that interacts only through conservative forces: ***Principle of mechanical energy conservation***

$$E_i = E_f$$

$$K_i + \sum U_i = K_f + \sum U_f$$

Example 8.3

A ball of mass *m* is attached to a light cord of length L, making up a pendulum. The ball is released from rest when the cord makes an angle θ_A with the vertical, and the pivoting point P is frictionless. Find the speed of the ball when it is at the lowest point, B.



Compute the potential energy at the maximum height, *h*. Remember where the 0 is.

$$h = L - L\cos\theta_A$$
$$U_i = mgL(1-\cos\theta_A)$$

Using the principle of mechanical energy conservation

$$K_i + U_i = K_f + U_f$$
$$mgh = mgL(1-\cos\theta_A) = \frac{1}{2}mv^2$$
$$v^2 = 2gL(1-\cos\theta_A)$$
$$\therefore v = \sqrt{2gL(1-\cos\theta_A)}$$

b) Determine tension T at the point B.

Recall the centripetal acceleration of a circular motion

$$F_r = ma_r = T - mg = m\frac{v^2}{L}$$
$$T = m\left(g + \frac{v^2}{L}\right) = m\left(g + \frac{2gL(1-\cos\theta_A)}{L}\right)$$
$$\therefore T = mg(3 - 2\cos\theta_A)$$

Cross check the result in a simple situation. What happens when the initial angle θ_A is 0? $T = mg$

Linear Momentum

The principle of energy conservation can be used to solve problems that are harder to solve just using Newton's laws. It is used to describe motion of an object or a system of objects.

A new concept of linear momentum can also be used to solve physical problems, especially the problems involving collisions of objects.

Linear momentum of an object whose mass is m and is moving at a velocity v is defined as

$$\vec{p} \equiv m\vec{v}$$

What can you tell from this definition about momentum?

1. *Momentum is a vector quantity.*
2. *The heavier the object the higher the momentum*
3. *The higher the velocity the higher the momentum*
4. *Its unit is kg.m/s*

What else can use see from the definition? Do you see force?

The change of momentum in a given time interval

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\left(m\vec{v}\right) = m\frac{d\vec{v}}{dt} = m\vec{a} = \vec{F}$$

Linear Momentum and Forces

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(m\vec{v}\right)$$

What can we learn from this Force-momentum relationship?

- *The rate of the change of particle's momentum is the same as the net force exerted on it.*
- *When net force is 0, the particle's linear momentum is constant.*
- *If a particle is isolated, the particle experiences no net force, therefore its momentum does not change and is conserved.*

Something else we can do with this relationship. What do you think it is?

The relationship can be used to study the case where the mass changes as a function of time.

Can you think of a few cases like this?

Motion of a meteorite

Trajectory a satellite

Conservation of Linear Momentum in a Two Particle System

Consider a system with two particles that does not have any external forces exerted on it. What is the impact of Newton's 3rd Law?

If particle#1 exerts force on particle #2, there must be another force that the particle #2 exerts on #1 as the reaction force. Both the forces are internal forces and the net force in the SYSTEM is still 0.

Now how would the momenta of these particles look like?

Let say that the particle #1 has momentum p_1 and #2 has p_2 at some point of time.

Using momentum-force relationship

$$\vec{F}_{21} = \frac{d\vec{p}_1}{dt} \quad \text{and} \quad \vec{F}_{12} = \frac{d\vec{p}_2}{dt}$$

And since net force of this system is 0

$$\sum \vec{F} = \vec{F}_{12} + \vec{F}_{21} = \frac{d\vec{p}_2}{dt} + \frac{d\vec{p}_1}{dt} = \frac{d}{dt}\left(\vec{p}_2 + \vec{p}_1\right) = 0$$

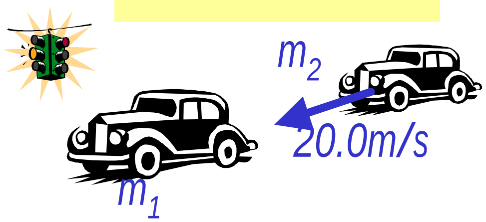
Therefore $\vec{p}_2 + \vec{p}_1 = const$

The total linear momentum of the system is conserved!!!

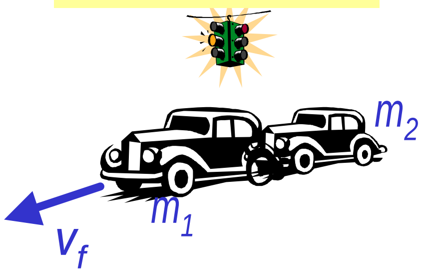
Example 9.5

A car of mass 1800kg stopped at a traffic light is rear-ended by a 900kg car, and the two become entangled. If the lighter car was moving at 20.0m/s before the collision what is the velocity of the entangled cars after the collision?

Before collision



After collision



The momenta before and after the collision are

$$\vec{p}_i = m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_2\vec{v}_{2i}$$
$$\vec{p}_f = m_1\vec{v}_{1f} + m_2\vec{v}_{2f} = (m_1 + m_2)\vec{v}_f$$

Since momentum of the system must be conserved

$$\vec{p}_i = \vec{p}_f$$
$$(m_1 + m_2)\vec{v}_f = m_2\vec{v}_{2i}$$
$$\vec{v}_f = \frac{m_2\vec{v}_{2i}}{(m_1 + m_2)} = \frac{900 \times 20.0\vec{i}}{900 + 1800} = 6.67\vec{i}\ m/s$$

What can we learn from these equations on the direction and magnitude of the velocity before and after the collision?

The cars are moving in the same direction as the lighter car's original direction to conserve momentum.

The magnitude is inversely proportional to its own mass.

Elastic and Inelastic Collisions

Momentum is conserved in any collisions as long as external forces negligible.

Collisions are classified as elastic or inelastic by the conservation of kinetic energy before and after the collisions.

Elastic Collision	A collision in which the total kinetic energy is the same before and after the collision.
Inelastic Collision	A collision in which the total kinetic energy is not the same before and after the collision.

Two types of inelastic collisions:Perfectly inelastic and inelastic

Perfectly Inelastic: Two objects stick together after the collision moving at a certain velocity together.

Inelastic: Colliding objects do not stick together after the collision but some kinetic energy is lost.

Note: Momentum is constant in all collisions but kinetic energy is only in elastic collisions.

Elastic and Perfectly Inelastic Collisions

In perfectly Inelastic collisions, the objects stick together after the collision, moving together. Momentum is conserved in this collision, so the final velocity of the stuck system is

$$m_1\overrightarrow{v_{1i}} + m_2\overrightarrow{v_{2i}} = (m_1 + m_2)\overrightarrow{v_f}$$

$$\overrightarrow{v_f} = \frac{m_1\overrightarrow{v_{1i}} + m_2\overrightarrow{v_{2i}}}{(m_1 + m_2)}$$

How about elastic collisions?

In elastic collisions, both the momentum and the kinetic energy are conserved. Therefore, the final speeds in an elastic collision can be obtained in terms of initial speeds as

$$m_1\overrightarrow{v_{1i}} + m_2\overrightarrow{v_{2i}} = m_1\overrightarrow{v_{1f}} + m_2\overrightarrow{v_{2f}}$$

$$\frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$$

$$m_1\left(v_{1i}^2 - v_{1f}^2\right) = m_2\left(v_{2i}^2 - v_{2f}^2\right)$$

$$m_1\left(v_{1i} - v_{1f}\right)\left(v_{1i} + v_{1f}\right) = m_2\left(v_{2i} - v_{2f}\right)\left(v_{2i} + v_{2f}\right)$$

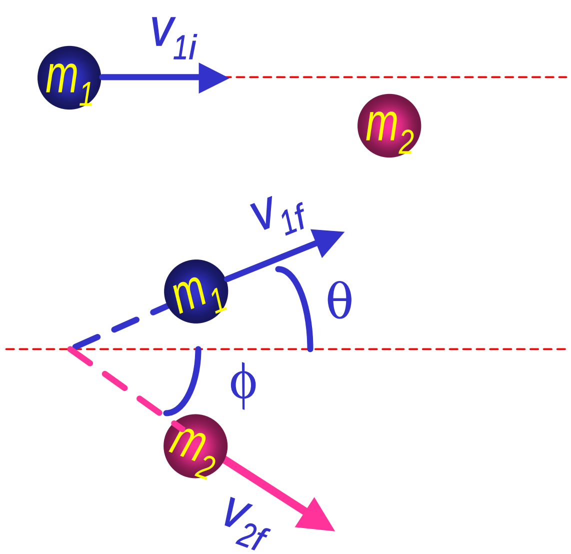
From 1-dim momentum conservation:

$$m_1\left(v_{1i} - v_{1f}\right) = m_2\left(v_{2i} - v_{2f}\right)$$

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i}; \quad v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} + \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{2i}$$

Example 9.9

Proton #1 with a speed 3.50x10^5 m/s collides elastically with proton #2 initially at rest. After the collision, proton #1 moves at an angle of 37^o to the horizontal axis and proton #2 deflects at an angle ϕ to the same axis. Find the final speeds of the two protons and the scattering angle of proton #2, ϕ.



Since both the particles are protons $m_1=m_2=m_p$.

Using momentum conservation, one obtains

$$m_p v_{1i} = m_p v_{1f} \cos q + m_p v_{2f} \cos f$$
$$m_p v_{1f} \sin q - m_p v_{2f} \sin f = 0$$

Canceling m_p and put in all known quantities, one obtains

$$v_{1f} \cos 37^\circ + v_{2f} \cos f = 3.50 \times 10^5 \quad (1)$$
$$v_{1f} \sin 37^\circ = v_{2f} \sin f \quad (2)$$

From kinetic energy conservation:

$$\left(3.50\times10^5\right)^2 = v_{1f}^2 + v_{2f}^2 \quad (3)$$

Solving Eqs. 1-3 equations, one gets

$$v_{1f} = 2.80 \times 10^5 \, m/s$$
$$v_{2f} = 2.11 \times 10^5 \, m/s$$
$$f = 53.0^\circ$$

Center of Mass of a Rigid Object

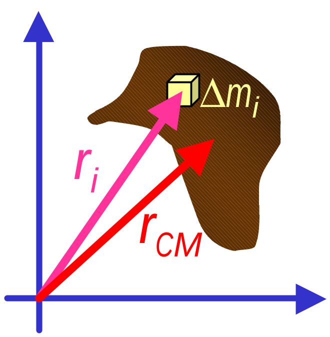
The formula for CM can be expanded to Rigid Object or a system of many particles

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_i m_ix_i}{\sum_i m_i}$$

$$y_{CM} = \frac{\sum_i m_iy_i}{\sum_i m_i}; \qquad z_{CM} = \frac{\sum_i m_iz_i}{\sum_i m_i}$$

The position vector of the center of mass of a many particle system is

$$\vec{r}_{CM} = x_{CM}\vec{i} + y_{CM}\vec{j} + z_{CM}\vec{k}$$
$$= \frac{\sum_i m_ix_i\vec{i} + \sum_i m_iy_i\vec{j} + \sum_i m_iz_i\vec{k}}{\sum_i m_i} = \frac{\sum_i m_i\vec{r}_i}{M}$$



A rigid body – an object with shape and size with mass spread throughout the body, ordinary objects – can be considered as a group of particles with mass m_i densely spread throughout the given shape of the object

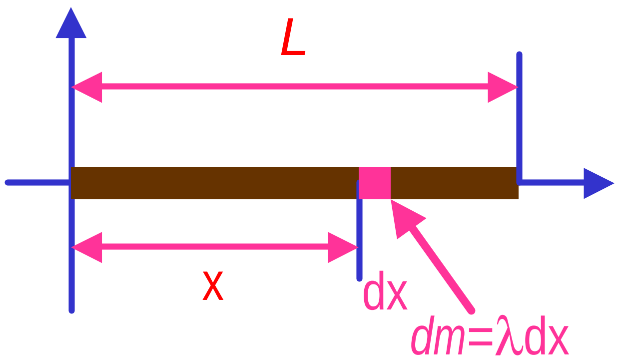
$$x_{CM} \approx \frac{\sum_i \Delta m_ix_i}{M}$$

$$x_{CM} = \lim_{\Delta m_i \to 0} \frac{\sum_i \Delta m_ix_i}{M} = \frac{1}{M}\int xdm$$

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}\,dm$$

Example 9.13

Show that the center of mass of a rod of mass *M* and length *L* lies in midway between its ends, assuming the rod has a uniform mass per unit length.



The formula for CM of a continuous object is

$$x_{CM} = \frac{1}{M}\int_{x=0}^{x=L} x\,dm$$

Since the density of the rod is constant, one can write

$$dm = \lambda\,dx; \text{ where } \lambda = M/L$$

Therefore

$$x_{CM} = \frac{1}{M}\int_{x=0}^{x=L} \lambda x\,dx = \frac{1}{M}\left[\frac{1}{2}\lambda x^2\right]_{x=0}^{x=L} = \frac{1}{M}\left(\frac{1}{2}\lambda L^2\right) = \frac{1}{M}\left(\frac{1}{2}ML\right) = \frac{L}{2}$$

Find the CM when the density of the rod non-uniform but varies linearly as a function of x, λ=α x

$$M = \int_{x=0}^{x=L} \lambda\,dx = \int_{x=0}^{x=L} \alpha x\,dx$$
$$= \left[\frac{1}{2}\alpha x^2\right]_{x=0}^{x=L} = \frac{1}{2}\alpha L^2$$

$$x_{CM} = \frac{1}{M}\int_{x=0}^{x=L} \lambda x\,dx = \frac{1}{M}\int_{x=0}^{x=L} \alpha x^2\,dx = \frac{1}{M}\left[\frac{1}{3}\alpha x^3\right]_{x=0}^{x=L}$$
$$= \frac{1}{M}\left(\frac{1}{3}\alpha L^3\right) = \frac{1}{M}\left(\frac{2}{3}ML\right) = \frac{2L}{3}$$

Motion of a Group of Particles

We've learned that the CM of a system can represent the motion of a system. Therefore, for an isolated system of many particles in which the total mass *M* is preserved, the velocity, total momentum, acceleration of the system are

Velocity of the system

$$\vec{v}_{CM}=\frac{d\vec{r}_{CM}}{dt}=\frac{d}{dt}\left(\frac{1}{M}\sum m_i\vec{r}_i\right)=\frac{1}{M}\sum m_i\frac{d\vec{r}_i}{dt}=\frac{\sum m_i\vec{v}_i}{M}$$

Total Momentum of the system

$$\vec{p}_{tot}=M\vec{v}_{CM}=M\frac{\sum m_i\vec{v}_i}{M}=\sum m_i\vec{v}_i=\sum \vec{p}_i$$

Acceleration of the system

$$\vec{a}_{CM}=\frac{d\vec{v}_{CM}}{dt}=\frac{d}{dt}\left(\frac{1}{M}\sum m_i\vec{v}_i\right)=\frac{1}{M}\sum m_i\frac{d\vec{v}_i}{dt}=\frac{\sum m_i\vec{a}_i}{M}$$

External force exerting on the system

$$\sum \vec{F}_{ext}=M\vec{a}_{CM}=\sum m_i\vec{a}_i=\frac{d\vec{p}_{tot}}{dt}$$

What about the internal forces?

If net external force is 0

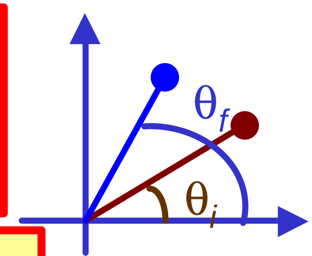
$$\sum \vec{F}_{ext}=0=\frac{d\vec{p}_{tot}}{dt};\quad \vec{p}_{tot}=\text{const}$$

System's momentum is conserved.

Angular Displacement, Velocity, and Acceleration

Using what we have learned in the previous slide, how would you define the angular displacement?

$$\Delta q = q_f - q_i$$



How about the average angular speed?

$$\overline{w} \equiv \frac{q_f - q_i}{t_f - t_i} = \frac{\Delta q}{\Delta t}$$

And the instantaneous angular speed?

$$w \equiv \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t} = \frac{dq}{dt}$$

By the same token, the average angular acceleration

$$\overline{a} \equiv \frac{w_f - w_i}{t_f - t_i} = \frac{\Delta w}{\Delta t}$$

And the instantaneous angular acceleration?

$$a \equiv \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = \frac{dw}{dt}$$

When rotating about a fixed axis, every particle on a rigid object rotates through the same angle and has the same angular speed and angular acceleration.

Rotational Kinematics

The first type of motion we have learned in linear kinematics was under a constant acceleration. We will learn about the rotational motion under constant acceleration, because these are the simplest motions in both cases.

Just like the case in linear motion, one can obtain

Angular Speed under constant angular acceleration:

$$w_f = w_i + a t$$

Angular displacement under constant angular acceleration:

$$q_f = q_i + w_i t + \frac{1}{2} a t^2$$

One can also obtain

$$w_f^2 = w_i^2 + 2a\left(q_f - q_i\right)$$

Example 10.1

A wheel rotates with a constant angular acceleration pf 3.50 rad/s^2. If the angular speed of the wheel is 2.00 rad/s at t_i=0, a) through what angle does the wheel rotate in 2.00s?

Using the angular displacement formula in the previous slide, one gets

$$\theta_f - \theta_i = \omega t + \frac{1}{2}\alpha t^2$$
$$= 2.00 \times 2.00 + \frac{1}{2}3.50 \times (2.00)^2$$
$$= 11.0\,rad \quad = \frac{11.0}{2\pi} rev. = 1.75\,rev.$$

What is the angular speed at t=2.00s?

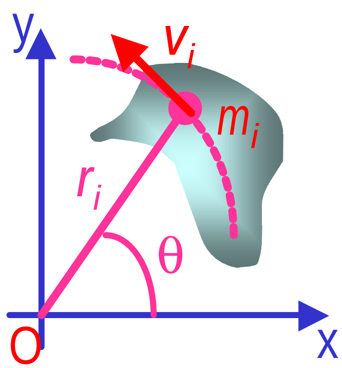
Using the angular speed and acceleration relationship

$$\omega_f = \omega_i + \alpha t$$
$$= 2.00 + 3.50 \times 2.00$$
$$= 9.00\,rad/s$$

Find the angle through which the wheel rotates between t=2.00 s and t=3.00 s.

$$\theta_2 = 11.0\,rad$$
$$\theta_3 = 2.00 \times 3.00 + \frac{1}{2}3.50 \times (3.00)^2 = 21.8\,rad$$
$$\Delta\theta = \theta_3 - \theta_2 = 10.8\,rad \quad = \frac{10.8}{2\pi} rev. = 1.72\,rev.$$

Rotational Energy



What do you think the kinetic energy of a rigid object that is undergoing a circular motion is?

Kinetic energy of a masslet, m_i, moving at a tangential speed, v_i, is

$$K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 w^2$$

Since a rigid body is a collection of masslets, the total kinetic energy of the rigid object is

$$K_R = \sum_i K_i = \frac{1}{2} \sum_i m_i r_i^2 w^2 = \frac{1}{2} \left(\sum_i m_i r_i^2 \right) w^2$$

By defining a new quantity called, Moment of Inertia, *I*, as

$$I = \sum_i m_i r_i^2$$

The above expression is simplified as

$$K_R = \frac{1}{2} I w^2$$

What are the dimension and unit of Moment of Inertia?

$$kg \cdot m^2 \qquad \left[ML^2\right]$$

What do you think the moment of inertia is?

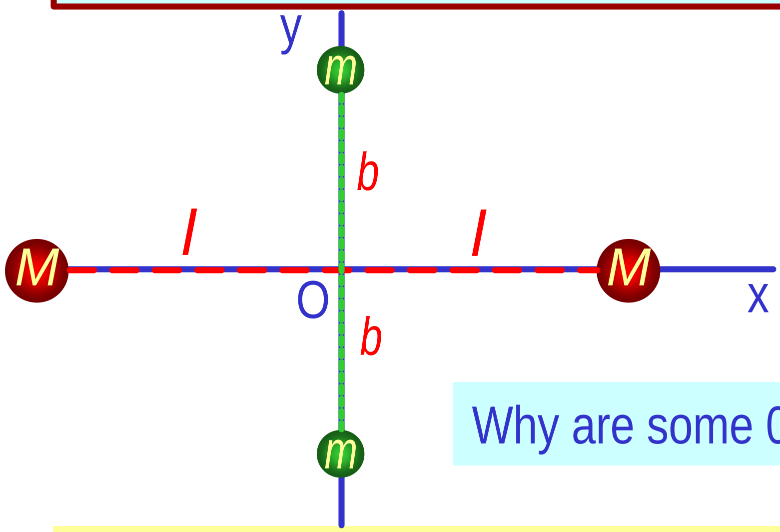
Measure of resistance of an object to changes in its rotational motion.

What similarity do you see between rotational and linear kinetic energies?

Mass and speed in linear kinetic energy are replaced by moment of inertia and angular speed.

Example 10.4

In a system consists of four small spheres as shown in the figure, assuming the radii are negligible and the rods connecting the particles are massless, compute the moment of inertia and the rotational kinetic energy when the system rotates about the y-axis at ω.



Since the rotation is about y axis, the moment of inertia about y axis, I_y, is

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + m \cdot 0^2 + m \cdot 0^2 = 2Ml^2$$

Why are some 0s?

This is because the rotation is done about y axis, and the radii of the spheres are negligible.

Thus, the rotational kinetic energy is

$$K_R = \frac{1}{2} I w^2 = \frac{1}{2}\left(2Ml^2\right)w^2 = Ml^2 w^2$$

Find the moment of inertia and rotational kinetic energy when the system rotates on the x-y plane about the z-axis that goes through the origin O.

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + mb^2 + mb^2 = 2\left(Ml^2 + mb^2\right)$$

$$K_R = \frac{1}{2} I w^2 = \frac{1}{2}\left(2Ml^2 + 2mb^2\right)w^2 = \left(Ml^2 + mb^2\right)w^2$$

Calculation of Moments of Inertia

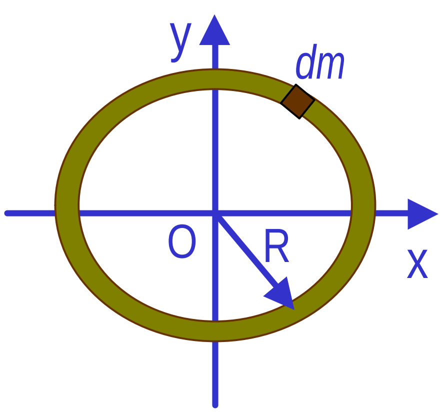
Moments of inertia for large objects can be computed, if we assume the object consists of small volume elements with mass, Δm_i.

The moment of inertia for the large rigid object $I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$

It is sometimes easier to compute moments of inertia in terms of volume of the elements rather than their mass

Using the volume density, ρ, replace d*m* in the above equation with dV. $\rho = \frac{dm}{dV};\ dm = \rho dV$ The moments of inertia becomes $I = \int \rho r^2 dV$

Example 10.5: Find the moment of inertia of a uniform hoop of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center.



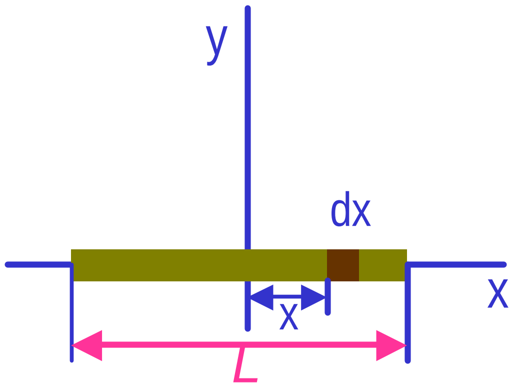
The moment of inertia is $I = \int r^2 dm = R^2 \int dm = MR^2$

What do you notice from this result?

The moment of inertia for this object is the same as that of a point of mass M at the distance R.

Example 10.6

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod and passing through its center of mass.



The line density of the rod is $l = \frac{M}{L}$

so the masslet is $dm = l\,dx = \frac{M}{L}dx$

The moment of inertia is

$$I = \int r^2 dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L}dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

What is the moment of inertia when the rotational axis is at one end of the rod.

$$I = \int r^2 dm = \int_0^L \frac{x^2 M}{L}dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_0^L$$

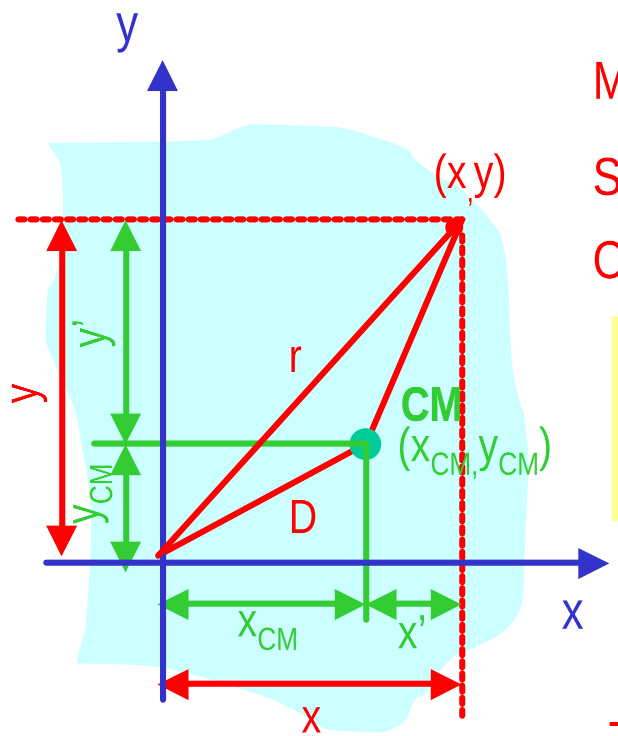
$$= \frac{M}{3L}\left[(L)^3 - 0\right] = \frac{M}{3L}\left(L^3\right) = \frac{ML^2}{3}$$

Will this be the same as the above. Why or why not?

Since the moment of inertia is resistance to motion, it makes perfect sense for it to be harder to move when it is rotating about the axis at one end.

Parallel Axis Theorem

Moments of inertia for highly symmetric object is relatively easy if the rotational axis is the same as the axis of symmetry. However if the axis of rotation does not coincide with axis of symmetry, the calculation can still be done in simple manner using **parallel-axis theorem**. $I = I_{CM} + MD^2$



Moment of inertia is defined $I = \int r^2 dm = \int \sqrt{\left(x^2 + y^2\right)} dm \quad (1)$

Since x and y are $x = x_{CM} + x'; \quad y = y_{CM} + y'$

One can substitute x and y in Eq. 1 to obtain

$$I = \int \left[\left(x_{CM} + x'\right)^2 + \left(y_{CM} + y'\right)^2\right] dm$$
$$= \left(x_{CM}^2 + y_{CM}^2\right)\int dm + 2x_{CM}\int x' dm + 2y_{CM}\int y' dm + \int\left(x'^2 + y'^2\right) dm$$

Since the x' and y' are the distance from CM, by definition

$$\int x' dm = 0$$
$$\int y' dm = 0$$

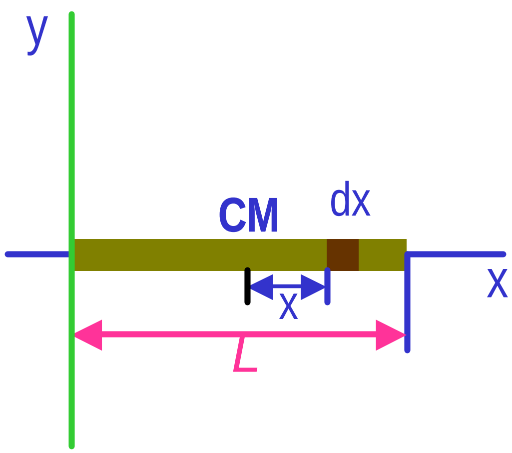
Therefore, the parallel-axis theorem $I = I_{CM} + MD^2$

What does this theorem tell you?

Moment of inertia of any object about any arbitrary axis are the same as the sum of moment of inertia for a rotation about the CM and that of the CM about the rotation axis.

Example 10.8

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis that goes through one end of the rod, using parallel-axis theorem.



The line density of the rod is $l = \frac{M}{L}$

so the masslet is $dm = l dx = \frac{M}{L} dx$

The moment of inertia about the CM

$$I_{CM} = \int r^2 dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

Using the parallel axis theorem

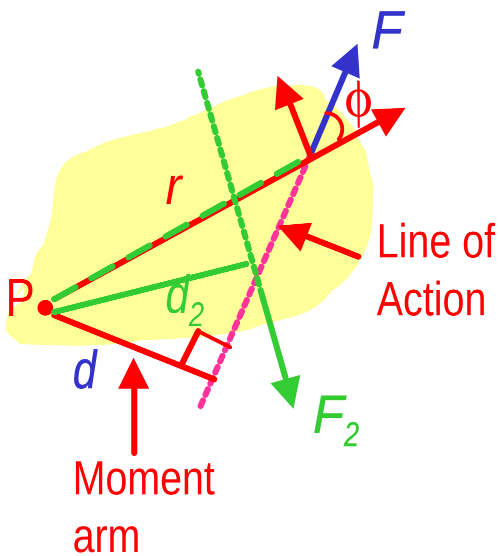
$$I = I_{CM} + D^2 M = \frac{ML^2}{12} + \left(\frac{L}{2}\right)^2 M = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}$$

The result is the same as using the definition of moment of inertia.

Parallel-axis theorem is useful to compute moment of inertia of a rotation of a rigid object with complicated shape about an arbitrary axis

Torque

Torque is the tendency of a force to rotate an object about some axis. Torque, t, is a vector quantity.



Consider an object pivoting about the point P by the force *F* being exerted at a distance r.

The line that extends out of the tail of the force vector is called the line of action.

The perpendicular distance from the pivoting point P to the line of action is called Moment arm.

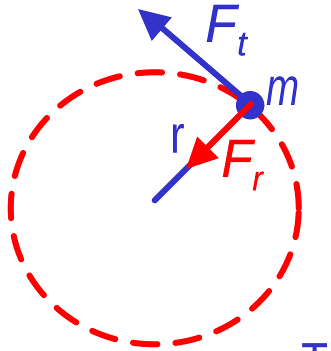
Magnitude of torque is defined as the product of the force exerted on the object to rotate it and the moment arm.

$$t \equiv rF\sin f = Fd$$

When there are more than one force being exerted on certain points of the object, one can sum up the torque generated by each force vectorially. The convention for sign of the torque is positive if rotation is in counter-clockwise and negative if clockwise.

$$\sum t = t_1 + t_2 = Fd - F_2 d_2$$

Torque & Angular Acceleration



Let's consider a point object with mass *m* rotating in a circle.

What forces do you see in this motion?

The tangential force F_t and radial force F_r

The tangential force F_t is $F_t = ma_t = mra$

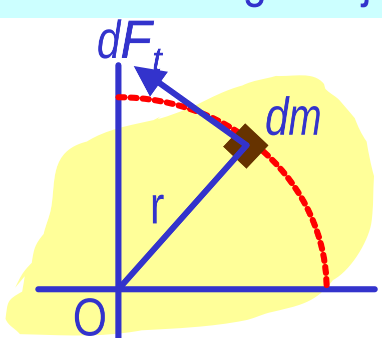
The torque due to tangential force F_t is $t = F_t r = ma_t r = mr^2 a$

What do you see from the above relationship? $t = Ia$

What does this mean? Torque acting on a particle is proportional to the angular acceleration.

What law do you see from this relationship? Analogs to Newton's 2nd law of motion in rotation.

How about a rigid object?



The external tangential force dF_t is $dF_t = dma_t = dmra$

The torque due to tangential force F_t is $dt = dF_t r = \left(r^2 dm\right)a$

The total torque is $\sum t = a\int r^2 dm = Ia$

What is the contribution due to radial force and why?

Contribution from radial force is 0, because its line of action passes through the pivoting point, making the moment arm 0.