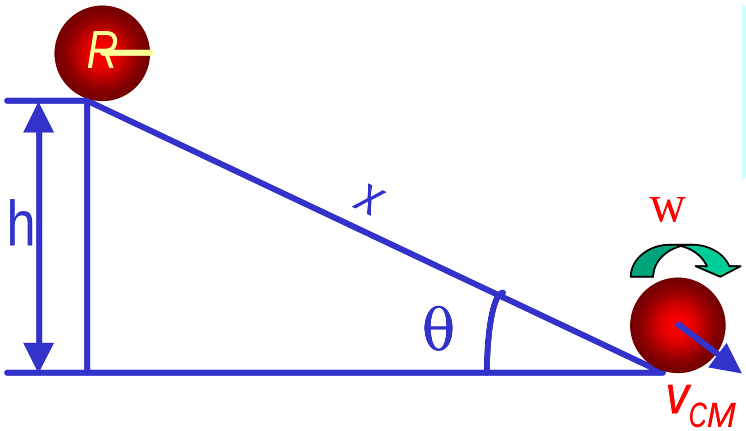
1443-501 Spring 2002
Lecture #16

Dr. Jaehoon Yu

1. Torque and Vector Product
2. Angular Momentum of a Particle
3. Angular Momentum of a Rotating Rigid Object
4. Conservation of Angular Momentum

Today's Homework Assignment is the Homework #7!!!

Kinetic Energy of a Rolling Sphere



Let's consider a sphere with radius R rolling down a hill without slipping.

$$K = \frac{1}{2} I_{CM} w^2 + \frac{1}{2} M R^2 w^2$$

Since $v_{CM}=R\omega$

$$K = \frac{1}{2} I_{CM}\left(\frac{v_{CM}}{R}\right)^2 + \frac{1}{2} M v_{CM}^2$$

$$= \frac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right) v_{CM}^2$$

What is the speed of the CM in terms of known quantities and how do you find this out?

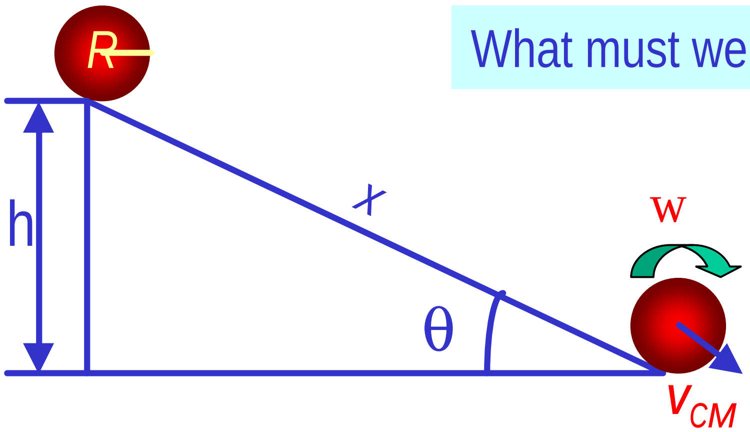
Since the kinetic energy at the bottom of the hill must be equal to the potential energy at the top of the hill

$$K = \frac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right) v_{CM}^2 = Mgh$$

$$v_{CM} = \sqrt{\frac{2gh}{1 + I_{CM}/MR^2}}$$

Example 11.1

For solid sphere as shown in the figure, calculate the linear speed of the CM at the bottom of the hill and the magnitude of linear acceleration of the CM.



What must we know first?

The moment of inertia the sphere with respect to the CM!!

$$I_{CM} = \int r^2 dm = \frac{2}{5} MR^2$$

Thus using the formula in the previous slide

$$v_{CM} = \sqrt{\frac{2gh}{1 + I_{CM} / MR^2}} = \sqrt{\frac{2gh}{1 + 2/5}} = \sqrt{\frac{10}{7} gh}$$

Since h=xsinθ, one obtains

$$v_{CM}^2 = \frac{10}{7} gx \sin q$$

Using kinematic relationship

$$v_{CM}^2 = 2 a_{CM} x$$

The linear acceleration of the CM is

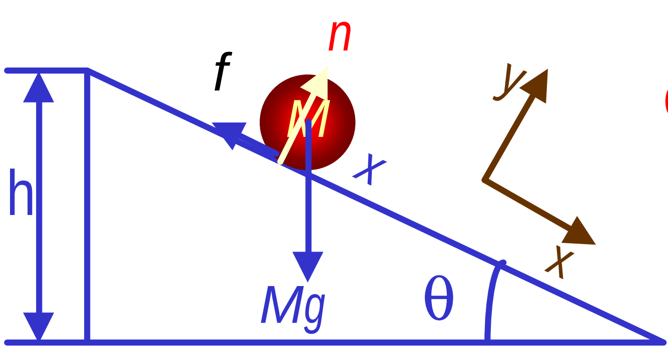
$$a_{CM} = \frac{v_{CM}^2}{2x} = \frac{5}{7} g \sin q$$

What do you see?

Linear acceleration of a sphere does not depend on anything but g and θ.

Example 11.2

For solid sphere as shown in the figure, calculate the linear speed of the CM at the bottom of the hill and the magnitude of linear acceleration of the CM. Solve this problem using Newton's second law, the dynamic method.



What are the forces involved in this motion?

Gravitational Force, Frictional Force, Normal Force

Newton's second law applied to the CM gives

$$\sum F_x = Mg\sin q - f = Ma_{CM}$$

$$\sum F_y = n - Mg\cos q = 0$$

Since the forces *Mg* and *n* go through the CM, their moment arm is 0 and do not contribute to torque, while the static friction ***f*** causes torque

$$t_{CM} = fR = I_{CM}a$$

We know that

$$I_{CM} = \frac{2}{5}MR^2$$

$$a_{CM} = Ra$$

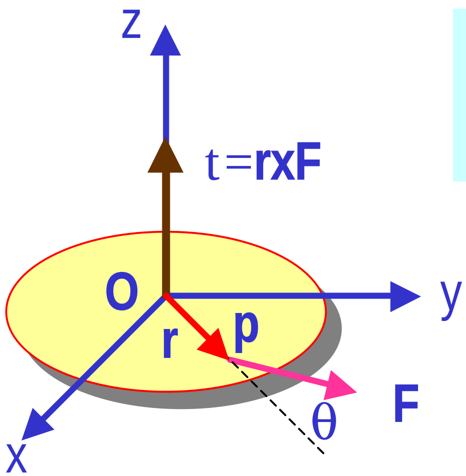
We obtain

$$f = \frac{I_{CM}a}{R} = \frac{\frac{2}{5}MR^2}{R}\left(\frac{a_{CM}}{R}\right) = \frac{2}{5}Ma_{CM}$$

Substituting ***f*** in dynamic equations

$$Mg\sin q = \frac{7}{5}Ma_{CM}\ ; \quad a_{CM} = \frac{5}{7}g\sin q$$

Torque and Vector Product



Let's consider a disk fixed onto the origin O and the force *F* exerts on the point p. What happens?

The disk will start rotating counter clockwise about the Z axis

The magnitude of torque given to the disk by the force *F* is

$$t = Fr\sin f$$

But torque is a vector quantity, what is the direction?
How is torque expressed mathematically?

$$\boxed{\vec{t} \equiv \vec{r} \times \vec{F}}$$

What is the direction?

The direction of the torque follows the right-hand rule!!

The above quantity is called
Vector product or Cross product

$$\vec{C} \equiv \vec{A} \times \vec{B}$$

$$\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin q$$

What is the result of a vector product?

Another vector

What is another vector operation we've learned?

Scalar product

$$C \equiv \vec{A} \cdot \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos q$$

Result? A scalar

Properties of Vector Product

Vector Product is Non-commutative

What does this mean?

If the order of operation changes the result changes $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$

Following the right-hand rule, the direction changes $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

Vector Product of a two parallel vectors is 0.

$$\left|\vec{C}\right| = \left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin q = \left|\vec{A}\right|\left|\vec{B}\right|\sin 0 = 0$$

Thus, $\vec{A}\times\vec{A} = 0$

If two vectors are perpendicular to each other

$$\left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin q = \left|\vec{A}\right|\left|\vec{B}\right|\sin 90^\circ = \left|\vec{A}\right|\left|\vec{B}\right| = AB$$

Vector product follows distribution law

$$\vec{A}\times\left(\vec{B}+\vec{C}\right) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$$

The derivative of a Vector product with respect to a scalar variable is

$$\frac{d\left(\vec{A}\times\vec{B}\right)}{dt} = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$$

More Properties of Vector Product

The relationship between unit vectors, $\vec{i}, \vec{j}$, and $\vec{k}$

$$\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=0$$

$$\vec{i}\times\vec{j}=-\vec{j}\times\vec{i}=\vec{k}$$

$$\vec{j}\times\vec{k}=-\vec{k}\times\vec{j}=\vec{i}$$

$$\vec{k}\times\vec{i}=-\vec{i}\times\vec{k}=\vec{j}$$

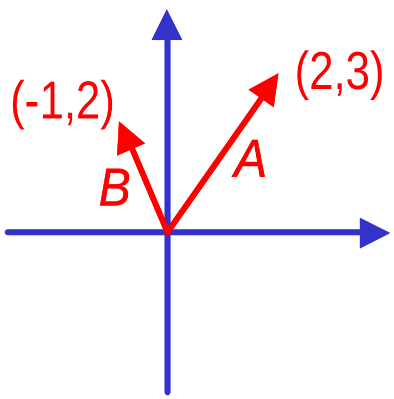
Vector product of two vectors can be expressed in the following determinant form

$$\vec{A}\times\vec{B}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\vec{i}\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}-\vec{j}\begin{vmatrix}A_x & A_z\\ B_x & B_z\end{vmatrix}+\vec{k}\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}$$

$$=\left(A_yB_z-A_zB_y\right)\vec{i}-\left(A_xB_z-A_zB_x\right)\vec{j}+\left(A_xB_y-A_yB_x\right)\vec{k}$$

Example 11.3

Two vectors lying in the xy plane are given by the equations *A=2i+3j* and *B=-i+2j*, verify that *AxB= -BxA*



$$\vec{A}\times\vec{B}=\left(2\vec{i}+3\vec{j}\right)\times\left(-\vec{i}+2\vec{j}\right)=4\vec{i}\times\vec{j}-3\vec{j}\times\vec{i}$$

Since $\vec{i}\times\vec{j}=\vec{k}$

$$\vec{A}\times\vec{B}=4\vec{k}+3\vec{j}\times\left(-\vec{i}\right)=4\vec{k}+3\vec{k}=7\vec{k}$$

$$\vec{B}\times\vec{A}=\left(-\vec{i}+2\vec{j}\right)\times\left(2\vec{i}+3\vec{j}\right)=-3\vec{i}\times\vec{j}+4\vec{j}\times\vec{i}$$

Using the same unit vector relationship as above

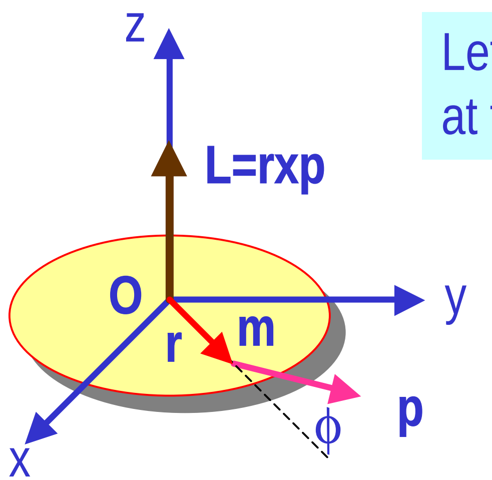
$$\vec{B}\times\vec{A}=-3\vec{k}-4\vec{k}=-7\vec{k}$$

Therefore, *AxB= -BxA*

Now prove this using determinant method

Angular Momentum of a Particle

If you grab onto a pole while running, your body will rotate about the pole, gaining angular momentum. We've used linear momentum to solve physical problems with linear motions, angular momentum will do the same for rotational motions.



Let's consider a point-like object (particle) with mass *m* located at the vector location *r* and moving with linear velocity *v*

The instantaneous angular momentum *L* of this particle relative to origin O is

$$\boxed{\vec{L} \equiv \vec{r} \times \vec{p}}$$

What is the unit and dimension of angular momentum? $kg \cdot m^2 / s^2$

Note that *L* depends on origin O. Why? Because *r* changes

What else do you learn? The direction of *L* is +z

Since *p* is *mv*, the magnitude of *L* becomes $L = mvr \sin f$

What do you learn from this?

If the direction of linear velocity points to the origin of rotation, the particle does not have any angular momentum.

If the linear velocity is perpendicular to position vector, the particle moves exactly the same way as a point on a rim.

The point O has to be inertial.

Angular Momentum and Torque

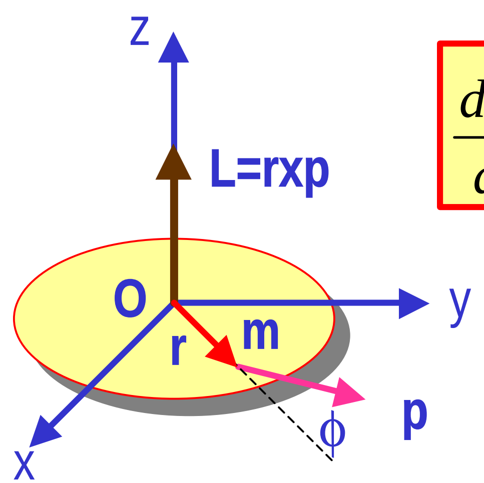
Can you remember how net force exerting on a particle and the change of its linear momentum are related?

$$\boxed{\sum \vec{F} = \frac{d\vec{p}}{dt}}$$

Total external forces exerting on a particle is the same as the change of its linear momentum.

The same analogy works in rotational motion between torque and angular momentum.

Net torque acting on a particle is $\boxed{\sum \vec{t} = \vec{r} \times \sum \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}}$



$$\boxed{\frac{d\vec{L}}{dt} = \frac{d\left(\vec{r} \times \vec{p}\right)}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = 0 + \vec{r} \times \frac{d\vec{p}}{dt}}$$

Why does this work?

Because *v* is parallel to the linear momentum

Thus the torque-angular momentum relationship

$$\boxed{\sum \vec{t} = \frac{d\vec{L}}{dt}}$$

The net torque acting on a particle is the same as the time rate change of its angular momentum

Angular Momentum of a System of Particles

The total angular momentum of a system of particles about some point is the vector sum of the angular momenta of the individual particles

$$\vec{L} = \vec{L_1} + \vec{L_2} + \ldots\ldots + \vec{L_n} = \sum \vec{L}$$

Since the individual angular momentum can change, the total angular momentum of the system can change.

Both internal and external forces can provide torque to individual particles. However, the internal forces do not generate net torque due to Newton's third law.

Let's consider a two particle system where the two exert forces on each other.

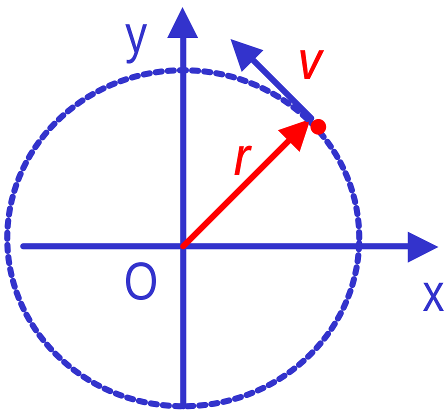
Since these forces are action and reaction forces with directions lie on the line connecting the two particles, the vector sum of the torque from these two becomes 0.

Thus the time rate change of the angular momentum of a system of particles is equal to the net external torque acting on the system

$$\boxed{\sum \vec{t}_{ext} = \frac{d\vec{L}}{dt}}$$

Example 11.4

A particle of mass *m* is moving in the xy plane in a circular path of radius r and linear velocity v about the origin O. Find the magnitude and direction of angular momentum with respect to O.



Using the definition of angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m\vec{r} \times \vec{v}$$

Since both the vectors, r and v, are on x-y plane and using right-hand rule, the direction of the angular momentum vector is +z (coming out of the screen)

The magnitude of the angular momentum is $\left|\vec{L}\right| = \left|m\vec{r} \times \vec{v}\right| = mrv\sin\phi = mrv\sin 90^\circ = mrv$

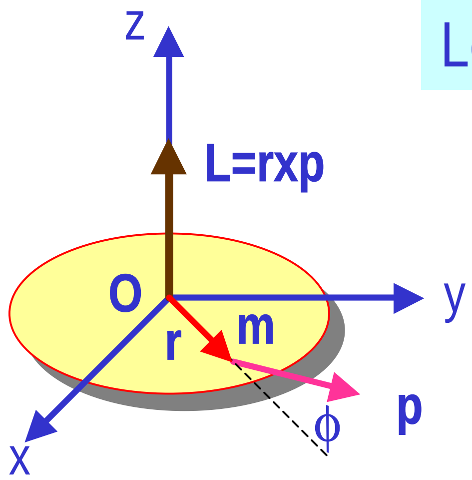
So the angular momentum vector can be expressed as $\vec{L} = mrv\vec{k}$

Find the angular momentum in terms of angular velocity ω.

Using the relationship between linear and angular speed

$$\vec{L} = mrv\vec{k} = mr^2\omega\vec{k} = mr^2\vec{\omega} = I\vec{\omega}$$

Angular Momentum of a Rotating Rigid Body



Let's consider a rigid body rotating about a fixed axis

Each particle of the object rotates in the xy plane about the z-axis at the same angular speed, ω

Magnitude of the angular momentum of a particle of mass m_i about origin O is $m_i v_i r_i$

$$L_i = m_i r_i v_i = m_i r_i^2 \omega$$

Summing over all particle's angular momentum about z axis

$$L_z = \sum_i L_i = \sum_i \left(m_i r_i^2 \omega \right)$$

What do you see?

$$L_z = \sum_i \left(m_i r_i^2 \right) \omega = I \omega$$

Since I is constant for a rigid body

$$\frac{dL_z}{dt} = I \frac{d\omega}{dt} = I \alpha$$

α is angular acceleration

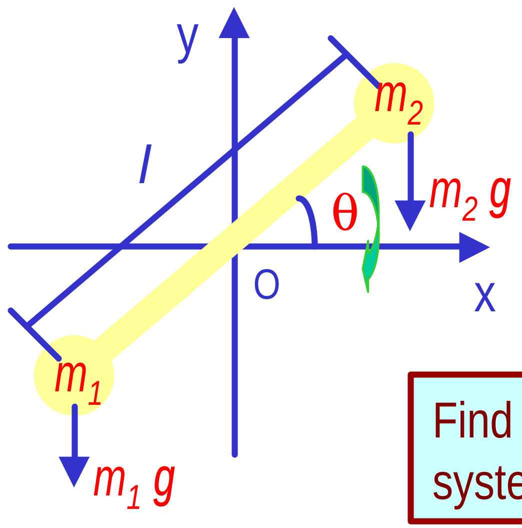
Thus the torque-angular momentum relationship becomes

$$\boxed{\sum \tau_{ext} = \frac{dL_z}{dt} = I \alpha}$$

Thus the net external torque acting on a rigid body rotating about a fixed axis is equal to the moment of inertia about that axis multiplied by the object's angular acceleration with respect to that axis.

Example 11.6

A rigid rod of mass *M* and length *l* pivoted without friction at its center. Two particles of mass m_1 and m_2 are connected to its ends. The combination rotates in a vertical plane with an angular speed of ω. Find an expression for the magnitude of the angular momentum.



The moment of inertia of this system is

$$I = I_{rod} + I_{m_1} + I_{m_2} = \frac{1}{12}Ml^2 + \frac{1}{4}m_1 l^2 + \frac{1}{4}m_2 l^2$$

$$= \frac{l^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)$$

$$\boxed{L = Iw = \frac{wl^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)}$$

Find an expression for the magnitude of the angular acceleration of the system when the rod makes an angle θ with the horizon.

If $m_1 = m_2$, no angular momentum because net torque is 0.

If θ=+/−π/2, at equilibrium so no angular momentum.

First compute net external torque

$$t_1 = m_1 g\frac{l}{2}\cos q;\; t_2 = -m_2 g\frac{l}{2}\cos q$$

$$t_{ext} = t_1 + t_2 = \frac{gl\cos q\left(m_1 - m_2\right)}{2}$$

Thus α becomes

$$a = \frac{\sum t_{ext}}{I} = \frac{\frac{1}{2}(m_1 - m_1)gl\cos q}{\frac{l^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)} = \frac{2(m_1 - m_1)\cos q}{\left(\frac{1}{3}M + m_1 + m_2\right)}g/l$$

Conservation of Angular Momentum

Remember under what condition the linear momentum is conserved?

Linear momentum is conserved when the net external force is 0.

$$\sum \vec{F} = 0 = \frac{d\vec{p}}{dt}$$
$$\vec{p} = const$$

By the same token, the angular momentum of a system is constant in both magnitude and direction, if the resultant external torque acting on the system is 0.

$$\sum \vec{t}_{ext} = \frac{d\vec{L}}{dt} = 0$$
$$\vec{L} = const$$

What does this mean?

Angular momentum of the system before and after a certain change is the same.

$$\vec{L}_i = \vec{L}_f = \text{constant}$$

Three important conservation laws for isolated system that does not get affected by external forces

$$\begin{cases} K_i + U_i = K_f + U_f & \textbf{Mechanical Energy} \\ \vec{p}_i = \vec{p}_f & \textbf{Linear Momentum} \\ \vec{L}_i = \vec{L}_f & \textbf{Angular Momentum} \end{cases}$$

Example 11.8

A start rotates with a period of 30days about an axis through its center. After the star undergoes a supernova explosion, the stellar core, which had a radius of 1.0x10^{4}km, collapses into a neutron start of radius 3.0km. Determine the period of rotation of the neutron star.

What is your guess about the answer?

The period will be significantly shorter, because its radius got smaller.

Let's make some assumptions:

1. There is no torque acting on it
2. The shape remains spherical
3. Its mass remains constant

Using angular momentum conservation

$$L_i = L_f$$
$$I_i w_i = I_f w_f$$

The angular speed of the star with the period T is

$$w = \frac{2p}{T}$$

Thus

$$w_f = \frac{I_i w_i}{I_f} = \frac{m r_i^2}{m r_f^2}\frac{2p}{T_i}$$

$$T_f = \frac{2p}{w_f} = \left(\frac{r_f^2}{r_i^2}\right) T_i = \left(\frac{3.0}{1.0\times 10^4}\right)^2 \times 30\, days = 2.7\times 10^{-6}\, days = 0.23\, s$$