1443-501 Spring 2002
Lecture #17

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1. Conditions for Equilibrium
2. Center of Gravity
3. Elastic Properties of Solids
   - Young's Modulus
   - Shear Modulus
   - Bulk Modulus

Today's Homework Assignment is the Homework #8!!!

2nd term exam on Wednesday, Apr. 10. Will cover chapters 10 -13.

Conditions for Equilibrium

What do you think does the term “An object is at its equilibrium” mean?

The object is either at rest (<u>Static Equilibrium</u>) or its center of mass is moving with a constant velocity (Dynamic Equilibrium).

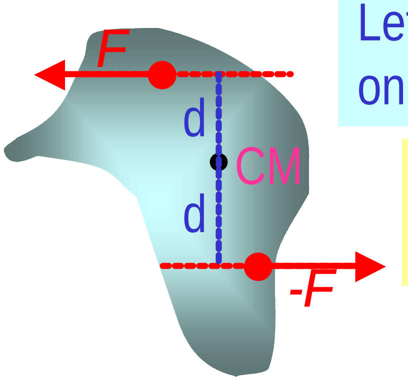
When do you think an object is at its equilibrium?

Translational Equilibrium: Equilibrium in linear motion

$$\sum \vec{F} = 0$$

Is this it?

The above condition is sufficient for a point-like particle to be at its static equilibrium. However for object with size this is not sufficient. One more condition is needed. What is it?



Let’s consider two forces equal magnitude but opposite direction acting on a rigid object as shown in the figure. What do you think will happen?

The object will rotate about the CM. The net torque acting on the object about any axis must be 0.

$$\sum \vec{t} = 0$$

For an object to be at its static equilibrium, the object should not have linear or angular speed.

$$v_{CM} = 0 \qquad w = 0$$

More on Conditions for Equilibrium

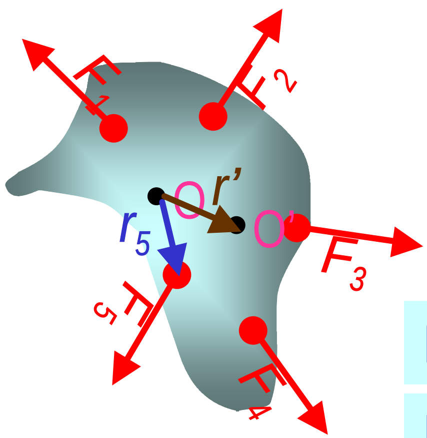
To simplify the problems, we will only deal with forces acting on x-y plane, giving torque only along z-axis. What do you think the conditions for equilibrium be in this case?

The six possible equations from the two vector equations turns to three equations.

$$\sum \vec{F} = 0 \qquad \sum F_x = 0 \qquad \sum F_y = 0$$

$$\sum \vec{\tau} = 0 \qquad \sum \tau_z = 0$$

What happens if there are many forces exerting on the object?



If an object is at its translational static equilibrium, and if the net torque acting on the object is 0 about one axis, the net torque must be 0 about any arbitrary axis.

Net Force exerting on the object

$$\sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots = 0$$

Net torque about O

$$\sum \vec{\tau}_O = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{r}_3 \times \vec{F}_3 + \cdots = \sum \vec{r}_i \times \vec{F}_i = 0$$

Position of force F_i about O'

$$\vec{r}_i\,' = \vec{r}_i - \vec{r}\,'$$

Net torque about O'

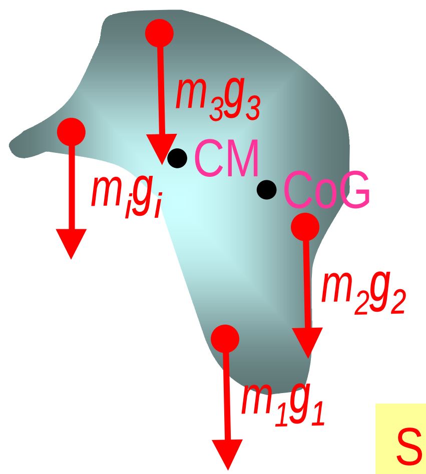
$$\sum \vec{\tau}_{O'} = \vec{r}\,'_1 \times \vec{F}_1 + \vec{r}\,'_2 \times \vec{F}_2 + \cdots = \left(\vec{r}_1 - \vec{r}\,'\right) \times \vec{F}_1 + \left(\vec{r}_2 - \vec{r}\,'\right) \times \vec{F}_2 + \cdots = \sum \vec{r}_i \times \vec{F}_i - \vec{r}\,' \times \sum \vec{F}_i$$

$$\sum \vec{\tau}_{O'} = \sum \vec{r}_i \times \vec{F}_i - \vec{r}\,' \times 0 = \sum \vec{\tau}_O = 0$$

Center of Gravity Revisited

When is the center of gravity of a rigid body the same as the center of mass?

Under the uniform gravitational field throughout the body of the object.



Let's consider an arbitrary shaped object

The center of mass of this object is

$$x_{CM} = \frac{\sum m_i x_i}{\sum m_i} = \frac{\sum m_i x_i}{M}$$

$$y_{CM} = \frac{\sum m_i y_i}{\sum m_i} = \frac{\sum m_i y_i}{M}$$

Let's now examine the case with gravitational acceleration on each point is g_i

Since the CoG is the point as if all the gravitational force is exerted on, the torque due to this force becomes

$$(m_1 g_1 + m_2 g_2 + \cdots) x_{CoG} = m_1 g_1 x_1 + m_2 g_2 x_2 + \cdots$$

Generalized expression for different *g* throughout the body

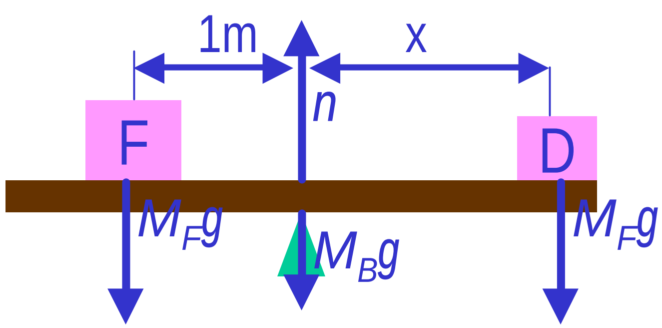
If *g* is uniform throughout the body

$$(m_1 + m_2 + \cdots) g x_{CoG} = (m_1 x_1 + m_2 x_2 + \cdots) g$$

$$x_{CoG} = \frac{\sum m_i x_i}{\sum m_i} = x_{CM}$$

Example 12.1

A uniform 40.0 N board supports a father and daughter weighing 800 N and 350 N, respectively. If the support (or fulcrum) is under the center of gravity of the board and the father is 1.00 m from CoG, what is the magnitude of normal force *n* exerted on the board by the support?



Since there is no linear motion, this system is in its translational equilibrium

$$\sum F_x = 0$$

$$\sum F_y = M_B g + M_F g + M_D g - n = 0$$

Therefore the magnitude of the normal force

$$n = 40.0 + 800 + 350 = 1190 N$$

Determine where the child should sit to balance the system.

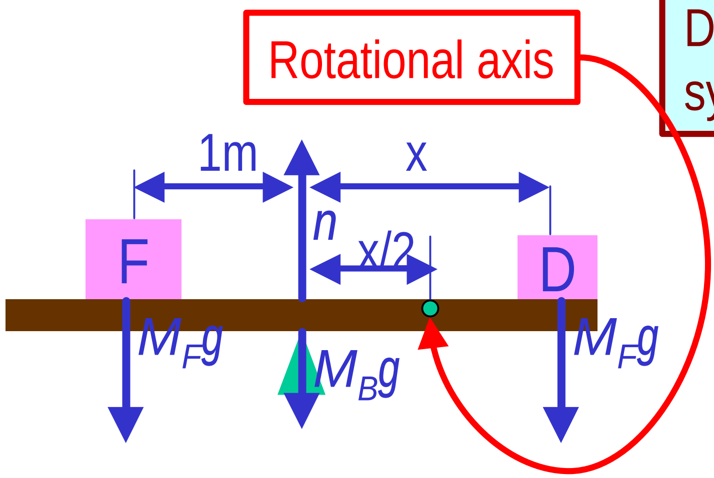
The net torque about the fulcrum by the three forces are

$$t = M_B g \cdot 0 + M_F g \cdot 1.00 - M_D g \cdot x = 0$$

Therefore to balance the system the daughter must sit

$$x = \frac{M_F g}{M_D g} \cdot 1.00m = \frac{800}{350} \cdot 1.00m = 2.29m$$

Example 12.1 Continued



Determine the position of the child to balance the system for different position of axis of rotation.

The net torque about the axis of rotation by all the forces are

$$\tau = M_B g \cdot x/2 + M_F g \cdot (1.00 + x/2) - n \cdot x/2 - M_D g \cdot x/2 = 0$$

Since the normal force is

$$n = M_B g + M_F g + M_D g$$

The net torque can be rewritten

$$\tau = M_B g \cdot x/2 + M_F g \cdot (1.00 + x/2) - (M_B g + M_F g + M_D g) \cdot x/2 - M_D g \cdot x/2 = M_F g \cdot 1.00 - M_D g \cdot x = 0$$

Therefore

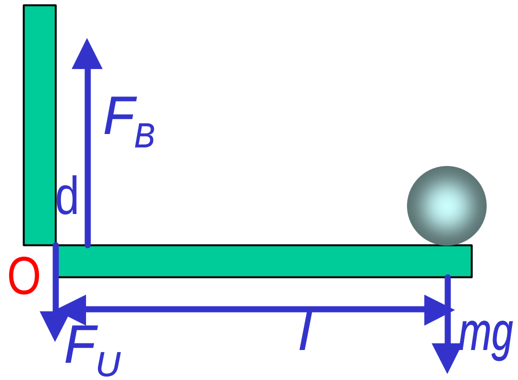
$$x = \frac{M_F g}{M_D g} \cdot 1.00m = \frac{800}{350} \cdot 1.00m = 2.29m$$

What do we learn?

No matter where the rotation axis is, net effect of the torque is identical.

Example 12.2

A person holds a 50.0N sphere in his hand. The forearm is horizontal. The biceps muscle is attached 3.00 cm from the joint, and the sphere is 35.0cm from the joint. Find the upward force exerted by the biceps on the forearm and the downward force exerted by the upper arm on the forearm and acting at the joint. Neglect the weight of forearm.



Since the system is in equilibrium, from the translational equilibrium condition

$$\sum F_x = 0$$

$$\sum F_y = F_B - F_U - mg = 0$$

From the rotational equilibrium condition

$$\sum t = F_U \cdot 0 + F_B \cdot d - mg \cdot l = 0$$

Thus, the force exerted by the biceps muscle is

$$F_B \cdot d = mg \cdot l$$

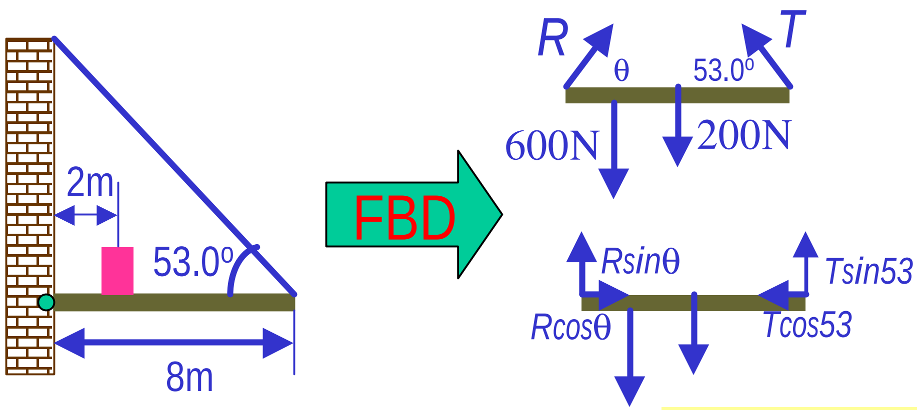
$$F_B = \frac{mg \cdot l}{d} = \frac{50.0 \times 35.0}{3.00} = 583N$$

Force exerted by the upper arm is

$$F_U = F_B - mg = 583 - 50.0 = 533N$$

Example 12.3

A uniform horizontal beam with a length of 8.00m and a weight of 200N is attached to a wall by a pin connection. Its far end is supported by a cable that makes an angle of 53.0º with the horizontal. If 600N person stands 2.00m from the wall, find the tension in the cable, as well as the magnitude and direction of the force exerted by the wall on the beam.



First the translational equilibrium, using components

$$\sum F_x = R\cos\theta - T\cos 53.0^\circ = 0$$

$$\sum F_y = R\sin\theta + T\sin 53.0^\circ - 600N - 200N = 0$$

From the rotational equilibrium

$$\sum \tau = T\sin 53.0^\circ \times 8.00 - 600N \times 2.00 - 200N \cdot 4.00m = 0$$

$$T = 313N$$

Using the translational equilibrium

$$R\cos\theta = T\cos 53.0^\circ$$

$$R\sin\theta = -T\sin 53.0^\circ + 600N + 200N$$

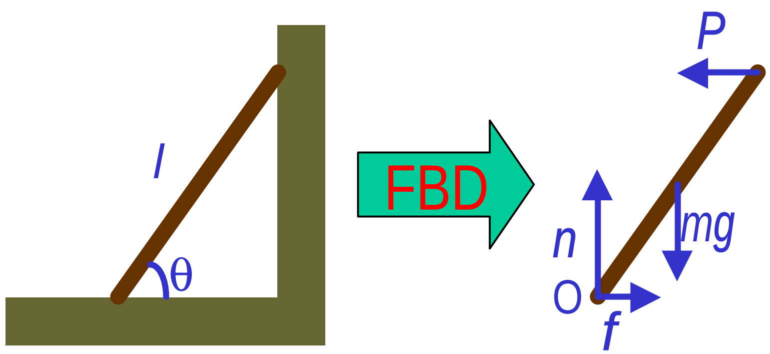
$$\theta = \tan^{-1}\left(\frac{800 - 313\times\sin 53.0^\circ}{313\cos 53.0^\circ}\right) = 71.7^\circ$$

And the magnitude of R is

$$R = \frac{T\cos 53.0^\circ}{\cos\theta} = \frac{313\times\cos 53.0^\circ}{\cos 71.1^\circ} = 582N$$

Example 12.4

A uniform ladder of length *l* and weight *mg*=50 N rests against a smooth, vertical wall. If the coefficient of static friction between the ladder and the ground is μ_s=0.40, find the minimum angle θ_{min} at which the ladder does not slip.



First the translational equilibrium, using components

$$\sum F_x = f - P = 0$$

$$\sum F_y = -mg + n = 0$$

Thus, the normal force is

$$n = mg = 50\,N$$

The maximum static friction force just before slipping is, therefore,

$$f_s^{max} = \mu_s n = 0.4 \times 50\,N = 20\,N = P$$

From the rotational equilibrium

$$\sum \tau_O = -mg\frac{l}{2}\cos\theta_{min} + Pl\sin\theta_{min} = 0$$

$$\theta_{min} = \tan^{-1}\left(\frac{mg}{2P}\right) = \tan^{-1}\left(\frac{50N}{40N}\right) = 51^{\circ}$$

Elastic Properties of Solids

We have been assuming that the objects do not change their shapes when external forces are exerting on it. It this realistic?

No. In reality, the objects get deformed as external forces act on it, though the internal forces resist the deformation as it takes place.

Deformation of solids can be understood in terms of Stress and Strain

Stress: A quantity proportional to the force causing deformation.
Strain: Measure of degree of deformation

It is empirically known that for small stresses, strain is proportional to stress

The constants of proportionality are called Elastic Modulus

$$\text{Elastic Modulus} \equiv \frac{\text{stress}}{\text{strain}}$$

Three types of Elastic Modulus

1. **Young's modulus**: Measure of the resistance in length
2. **Shear modulus**: Measure of the resistance in plane
3. **Bulk modulus**: Measure of the resistance in volume

Young's Modulus

Let's consider a long bar with cross sectional area A and initial length L_i.

Tensile stress $\text{Tensile Stress} \equiv \frac{F_{ex}}{A}$ Tensile strain $\text{Tensile Strain} \equiv \frac{\Delta L}{L_i}$

Young's Modulus is defined as

$$Y \equiv \frac{\text{Tensile Stress}}{\text{Tensile Strain}} = \frac{F_{ex}/A}{\Delta L / L_i}$$

Used to characterize a rod or wire stressed under tension or compression

What is the unit of Young's Modulus?

Force per unit area

Experimental Observations

1. For fixed external force, the change in length is proportional to the original length
2. The necessary force to produce a given strain is proportional to the cross sectional area

Elastic limit: Maximum stress that can be applied to the substance before it becomes permanently deformed

Shear Modulus

Another type of deformation occurs when an object is under a force tangential to one of its surfaces while the opposite face is held fixed by another force.

Shear stress

$$\text{Shear Stress} \equiv \frac{\text{Tangential Force}}{\text{Surface Area the force applies}} = \frac{F}{A}$$

Shear strain

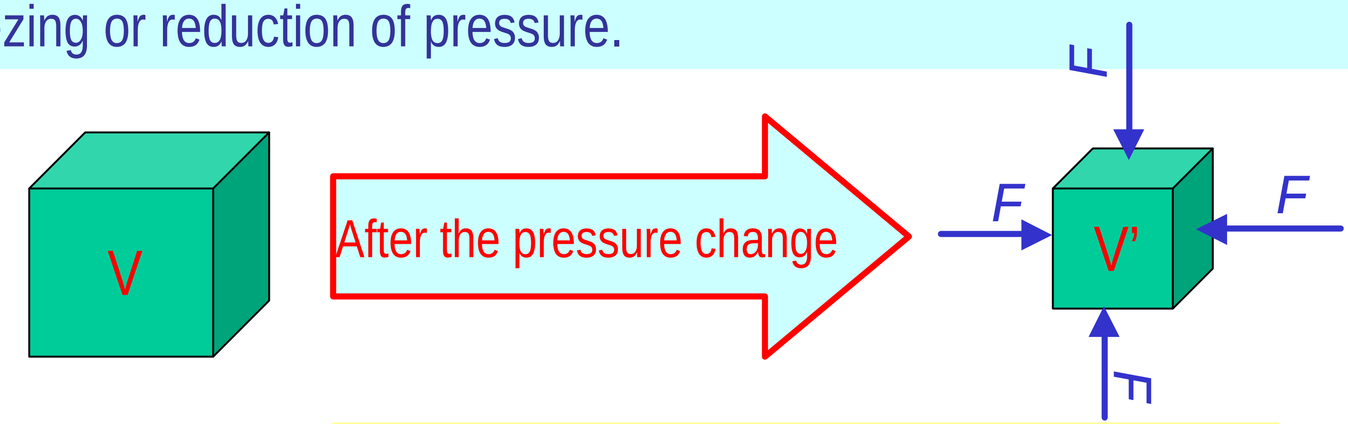
$$\text{Shear Strain} \equiv \frac{\Delta x}{h}$$

Shear Modulus is defined as

$$S \equiv \frac{\text{Shear Stress}}{\text{Shear Strain}} = \frac{F/A}{\Delta x/h}$$

Bulk Modulus

Bulk Modulus characterizes the response of a substance to uniform squeezing or reduction of pressure.



Volume stress =pressure

$$\text{Pressure} \equiv \frac{\text{Normal Force}}{\text{Surface Area the force applies}} = \frac{F}{A}$$

If the pressure on an object changes by ΔP=ΔF/A, the object will undergo a volume change ΔV.

Bulk Modulus is defined as

$$B \equiv \frac{\text{Volume Stress}}{\text{Volume Strain}} = -\frac{\Delta F / A}{\Delta V / V_i} = -\frac{\Delta P}{\Delta V / V_i}$$

Because the change of volume is reverse to change of pressure.

Compressibility is the reciprocal of Bulk Modulus

Example 12.7

A solid brass sphere is initially under normal atmospheric pressure of $1.0\times10^5 N/m^2$. The sphere is lowered into the ocean to a depth at which the pressures is $2.0\times10^7 N/m^2$. The volume of the sphere in air is $0.5m^3$. By how much its volume change once the sphere is submerged?

Since bulk modulus is $$B = -\frac{\Delta P}{\Delta V / V_i}$$

The amount of volume change is $$\Delta V = -\frac{\Delta P V_i}{B}$$

From table 12.1, bulk modulus of brass is 6.1×10^{10} N/m^2

The pressure change ΔP is $$\Delta P = P_f - P_i = 2.0\times10^7 - 1.0\times10^5 \approx 2.0\times10^7$$

Therefore the resulting volume change ΔV is $$\Delta V = V_f - V_i = -\frac{2.0\times10^7\times0.5}{6.1\times10^{10}} = -1.6\times10^{-4} m^3$$

The volume has decreased.