1443-501 Spring 2002
Lecture #18

Dr. Jaehoon Yu

1. Simple Harmonic Motion
2. Energy of the Simple Harmonic Oscillator
3. The Pendulum

Today's Homework Assignment is the Homework #9!!!

2nd term exam on Wednesday, Apr. 10. Will cover chapters 10 -13.

Simple Harmonic Motion

What do you think a harmonic motion is?

Motion that occurs by the force that depends on displacement, and the force is always directed toward the system's equilibrium position.

What is a system that has this kind of character?

A system consists of a mass and a spring

When a spring is stretched from its equilibrium position by a length x, the force acting on the mass is

$$F = -kx$$

It's negative, because the force resists against the change of length, directed toward the equilibrium position.

From Newton's second law $F = ma = -kx$ we obtain $a = -\frac{k}{m}x$

This is a second order differential equation that can be solved but it is beyond the scope of this class.

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Condition for simple harmonic motion

What do you observe from this equation?

Acceleration is proportional to displacement from the equilibrium
Acceleration is opposite direction to displacement

Equation of Simple Harmonic Motion

The solution for the 2nd order differential equation

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$x = A\cos(wt + f)$$

Amplitude

Phase

Angular Frequency

Phase constant

Generalized expression of a simple harmonic motion

Let's think about the meaning of this equation of motion

What happens when t=0 and ϕ=0?

$$x = A\cos(0+0) = A$$

What is ϕ if x is not A at t=0?

$$x = A\cos(f) = x'$$

$$f = \cos^{-1}(x')$$

What are the maximum/minimum possible values of x?

A/-A

An oscillation is fully characterized by its:

- Amplitude
- Period or frequency
- Phase constant

More on Equation of Simple Harmonic Motion

What is the time for full cycle of oscillation?

Since after a full cycle the position must be the same

$$x = A\cos(\omega(t+T)+\phi) = A\cos(\omega t + 2\pi + \phi)$$

The period

$$T = \frac{2\pi}{\omega}$$

One of the properties of an oscillatory motion

How many full cycles of oscillation does this undergo per unit time?

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

Frequency

What is the unit?

1/s=Hz

Let's now think about the object's speed and acceleration.

$$x = A\cos(\omega t + \phi)$$

Speed at any given time

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi)$$

Max speed

$$v_{\max} = \omega A$$

Acceleration at any given time

$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \phi) = -\omega^2 x$$

Max acceleration

$$a_{\max} = \omega^2 A$$

What do we learn about acceleration?

Acceleration is reverse direction to displacement

Acceleration and speed are $\pi/2$ off phase:

When *v* is maximum, *a* is at its minimum

Simple Harmonic Motion continued

This constant determines the starting position of a simple harmonic motion.

$$x = A\cos(wt + f)$$ At t=0 $$x\big|_{t=0} = A\cos f$$

This constant is important when there are more than one harmonic oscillation involved in the motion and to determine the overall effect of the composite motion

Let's determine phase constant and amplitude

At t=0 $$x_i = A\cos f \qquad v_i = -wA\sin f$$

By taking the ratio, one can obtain the phase constant

$$\boxed{f = \tan^{-1}\left(-\frac{x_i}{wv_i}\right)}$$

By squaring the two equation and adding them together, one can obtain the amplitude

$$x_i^2 = A^2\cos^2 f$$

$$v_i^2 = w^2A^2\sin^2 f$$

$$A^2\left(\cos^2 f + \cos^2 f\right) = A^2 = x_i^2 + \left(\frac{v_i}{w}\right)^2$$

$$\boxed{A = \sqrt{x_i^2 + \left(\frac{v_i}{w}\right)^2}}$$

Example 13.1

An object oscillates with simple harmonic motion along the x-axis. Its displacement from the origin varies with time according to the equation; $x = (4.00m)\cos\left(pt + \frac{p}{4}\right)$ where t is in seconds and the angles is in the parentheses are in radians. a) Determine the amplitude, frequency, and period of the motion.

From the equation of motion: $x = A\cos(wt + f) = (4.00m)\cos\left(pt + \frac{p}{4}\right)$

The amplitude, A, is $A = 4.00m$ The angular frequency, ω, is $w = p$

Therefore, frequency and period are

$$T = \frac{2p}{w} = \frac{2p}{p} = 2s$$

$$f = \frac{1}{T} = \frac{w}{2p} = \frac{p}{2p} = \frac{1}{2}s^{-1}$$

b)Calculate the velocity and acceleration of the object at any time t

Taking the first derivative on the equation of motion, the velocity is

$$v = \frac{dx}{dt} = -(4.00 \times p)\sin\left(pt + \frac{p}{4}\right)m/s$$

By the same token, taking the second derivative of equation of motion, the acceleration, a, is

$$a = \frac{d^2x}{dt^2} = -(4.00 \times p^2)\cos\left(pt + \frac{p}{4}\right)m/s^2$$

Fig13-10.ip

Simple Block-Spring System

A block attached at the end of a spring on a frictionless surface experiences acceleration when the spring is displaced from an equilibrium position.

$$a = -\frac{k}{m}x$$

This becomes a second order differential equation

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

If we denote

$$w^2 = \frac{k}{m}$$

The resulting differential equation becomes

$$\frac{d^2x}{dt^2} = -w^2 x$$

Since this satisfies condition for simple harmonic motion, we can take the solution

$$\boxed{x = A\cos(wt + f)}$$

Does this solution satisfy the differential equation?

Let's take derivatives with respect to time

$$\frac{dx}{dt} = A\frac{d}{dt}(\cos(wt+f)) = -w\mathrm{A}\sin(wt+f)$$

Now the second order derivative becomes

$$\frac{d^2x}{dt^2} = -w\mathrm{A}\frac{d}{dt}(\sin(wt+f)) = -w^2\mathrm{A}\cos(wt+f) = -w^2x$$

Whenever the force acting on a particle is linearly proportional to the displacement from some equilibrium position and in the opposite direction, the particle moves in simple harmonic motion.

More Simple Block-Spring System

How do the period and frequency of this harmonic motion look?

Since the angular frequency ω is $w = \sqrt{\frac{k}{m}}$

The period, T, becomes $T = \frac{2p}{w} = 2p\sqrt{\frac{m}{k}}$

So the frequency is $f = \frac{1}{T} = \frac{w}{2p} = \frac{1}{2p}\sqrt{\frac{k}{m}}$

What can we learn from these?

- Frequency and period do not depend on amplitude
- Period is inversely proportional to spring constant and proportional to mass

---

Special case #1

Let's consider that the spring is stretched to distance A and the block is let go from rest, giving 0 initial speed; x_i=A, v_i=0,

$$x = A\cos wt$$

$$v = \frac{dx}{dt} = -wA\sin wt$$

$$a = \frac{d^2x}{dt^2} = -w^2A\cos wt$$

$$a_i = -w^2A = -kA/m$$

This equation of motion satisfies all the conditions. So it is the solution for this motion.

Special case #2

Suppose block is given non-zero initial velocity v_i to positive x at the instant it is at the equilibrium, x_i=0

$$f = \tan^{-1}\left(-\frac{v_i}{wx_i}\right) = \tan^{-1}(-\infty) = -\frac{p}{2}$$

$$x = A\cos\left(wt - \frac{p}{2}\right) = A\sin(wt)$$

Is this a good solution?

Example 13.2

A car with a mass of 1300kg is constructed so that its frame is supported by four springs. Each spring has a force constant of 20,000N/m. If two peoploe riding in the car have a combined mass of 160kg, find the frequency of vibration of the car after it is driven over a pothole in the road.

Let's assume that mass is evenly distributed to all four springs.

The total mass of the system is 1460kg.

Therefore each spring supports 365kg each.

From the frequency relationship based on Hook's law

$$f = \frac{1}{T} = \frac{w}{2p} = \frac{1}{2p}\sqrt{\frac{k}{m}}$$

Thus the frequency for vibration of each spring is

$$f = \frac{1}{2p}\sqrt{\frac{k}{m}} = \frac{1}{2p}\sqrt{\frac{20000}{365}} = 1.18\ s^{-1} = 1.18\ Hz$$

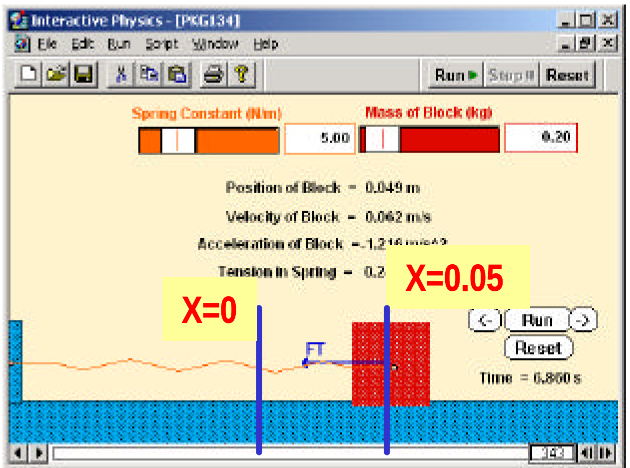
How long does it take for the car to complete two full vibrations?

The period is $T = \frac{1}{f} = 2p\sqrt{\frac{m}{k}} = 0.849\ s$ For two cycles $2T = 1.70\ s$

Ex13-03.ip

Example 13.3

A block with a mass of 200g is connected to a light spring for which the force constant is 5.00 N/m and is free to oscillate on a horizontal, frictionless surface. The block is displaced 5.00 cm from equilibrium and released from reset. Find the period of its motion.



From the Hook's law, we obtain

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5.00}{0.20}} = 5.00\ s^{-1}$$

As we know, period does not depend on the amplitude or phase constant of the oscillation, therefore the period, T, is simply

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5.00} = 1.26\ s$$

Determine the maximum speed of the block.

From the general expression of the simple harmonic motion, the speed is

$$v_{\max} = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$
$$= \omega A = 5.00 \times 0.05 = 0.25 m/s$$

Energy of the Simple Harmonic Oscillator

How do you think the mechanical energy of the harmonic oscillator look without friction?

Kinetic energy of a harmonic oscillator is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}mw^2A^2\sin^2(wt+f)$$

The elastic potential energy stored in the spring

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(wt+f)$$

Therefore the total mechanical energy of the harmonic oscillator is

$$E = K + U = \frac{1}{2}\left[mw^2A^2\sin^2(wt+f) + kA^2\cos^2(wt+f)\right]$$

Since $w = \sqrt{k/m}$

$$\boxed{E = K + U = \frac{1}{2}\left[kA^2\sin^2(wt+f) + kA^2\cos^2(wt+f)\right] = \frac{1}{2}kA^2}$$

Total mechanical energy of a simple harmonic oscillator is a constant of a motion and is proportional to the square of the amplitude

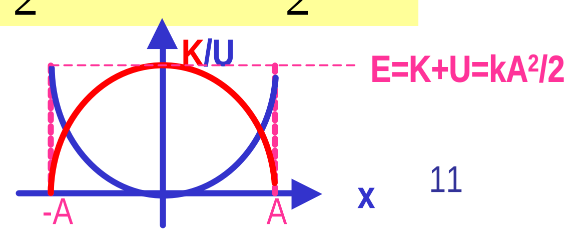
Maximum K is when U=0

$$K = \frac{1}{2}mv^2_{\max} = \frac{1}{2}mw^2A^2\sin^2(wt+f) = \frac{1}{2}mw^2A^2 = \frac{1}{2}kA^2$$

One can obtain speed

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

$$v = \pm\sqrt{k/m\left(A^2 - x^2\right)} = \pm w\sqrt{A^2 - x^2}$$



Ex13-04.ip

Example 13.4

A 0.500kg cube connected to a light spring for which the force constant is 20.0 N/m oscillates on a horizontal, frictionless track. a) Calculate the total energy of the system and the maximum speed of the cube if the amplitude of the motion is 3.00 cm.

From the problem statement, A and k are

$$k = 20.0 N/m$$
$$A = 3.00 cm = 0.03 m$$

The total energy of the cube is

$$E = K + U = \frac{1}{2}kA^2 = \frac{1}{2}20.0\times(0.03)^2 = 9.00\times10^{-3} J$$

Maximum speed occurs when kinetic energy is the same as the total energy

$$K = \frac{1}{2}mv_{max}^2 = E = \frac{1}{2}kA^2$$
$$v_{max} = A\sqrt{\frac{k}{m}} = 0.03\sqrt{\frac{20.0}{0.500}} = 0.190 m/s$$

b) What is the velocity of the cube when the displacement is 2.00 cm.

velocity at any given displacement is

$$v = \pm\sqrt{k/m\left(A^2 - x^2\right)} = \pm\sqrt{20.0\cdot\left(0.03^2 - 0.02^2\right)/0.500} = 0.141 m/s$$

c) Compute the kinetic and potential energies of the system when the displacement is 2.00 cm.

Kinetic energy, K

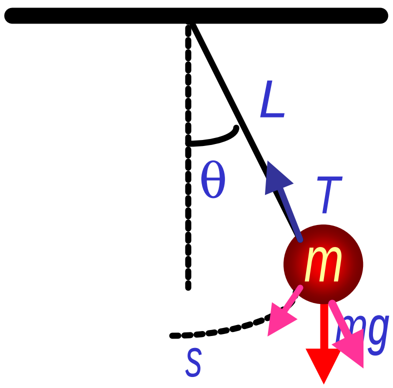
$$K = \frac{1}{2}mv^2 = \frac{1}{2}0.500\times(0.141)^2 = 4.97\times10^{-3} J$$

Potential energy, U

$$U = \frac{1}{2}kx^2 = \frac{1}{2}20.0\times(0.02)^2 = 4.00\times10^{-3} J$$

The Pendulum

A simple pendulum also performs periodic motion.



The net force exerted on the bob is

$$\sum F_r = T - mg\cos\theta_A = 0$$

$$\sum F_t = -mg\sin\theta_A = ma = m\frac{d^2s}{dt^2}$$

Since the arc length, s, is $s = L\theta_A$

$$\frac{d^2s}{dt^2} = L\frac{d^2\theta}{dt^2} = -g\sin\theta \quad \text{results} \Rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

Again became a second degree differential equation, satisfying conditions for simple harmonic motion

If θ is very small, sinθ~θ $\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta = -\omega^2\theta$ giving angular frequency $\omega = \sqrt{\frac{g}{L}}$

The period for this motion is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$

The period only depends on the length of the string and the gravitational acceleration

Example 13.5

Christian Huygens (1629-1695), the greatest clock maker in history, suggested that an international unit of length could be defined as the length of a simple pendulum having a period of exactly 1s. How much shorter would out length unit be had this suggestion been followed?

Since the period of a simple pendulum motion is

$$T = \frac{2p}{w} = 2p\sqrt{\frac{L}{g}}$$

The length of the pendulum in terms of T is

$$L = \frac{T^2 g}{4p^2}$$

Thus the length of the pendulum when T=1s is

$$L = \frac{T^2 g}{4p^2} = \frac{1\times 9.8}{4p^2} = 0.248\,m$$

Therefore the difference in length with respect to the current definition of 1m is

$$\Delta L = 1 - L = 1 - 0.248 = 0.752\,m$$