1443-501 Spring 2002
Lecture #19
Dr. Jaehoon Yu

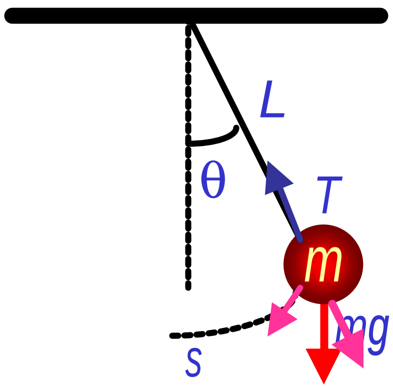
1. The Pendulum
2. Physical Pendulum
3. Simple Harmonic and Uniform Circular Motions
4. Damped Oscillation
5. Review Examples Ch. 10-13

No Homework Assignment today!!!!!

2nd term exam on Wednesday, Apr. 10. Will cover chapters 10 -13.

The Pendulum

A simple pendulum also performs periodic motion.



The net force exerted on the bob is

$$\sum F_r = T - mg\cos q_A = 0$$

$$\sum F_t = -mg\sin q_A = ma = m\frac{d^2s}{dt^2}$$

Since the arc length, s, is $s = Lq_A$

$$\frac{d^2s}{dt^2} = L\frac{d^2q}{dt^2} = -g\sin q$$

results

$$\frac{d^2q}{dt^2} = -\frac{g}{L}\sin q$$

Again became a second degree differential equation, satisfying conditions for simple harmonic motion

If θ is very small, sinθ~θ

$$\frac{d^2q}{dt^2} = -\frac{g}{L}q = -w^2q$$

giving angular frequency

$$w = \sqrt{\frac{g}{L}}$$

The period for this motion is

$$T = \frac{2p}{w} = 2p\sqrt{\frac{L}{g}}$$

The period only depends on the length of the string and the gravitational acceleration

Example 13.5

Christian Huygens (1629-1695), the greatest clock maker in history, suggested that an international unit of length could be defined as the length of a simple pendulum having a period of exactly 1s. How much shorter would out length unit be had this suggestion been followed?

Since the period of a simple pendulum motion is

$$T = \frac{2p}{w} = 2p\sqrt{\frac{L}{g}}$$

The length of the pendulum in terms of T is

$$L = \frac{T^2 g}{4p^2}$$

Thus the length of the pendulum when T=1s is

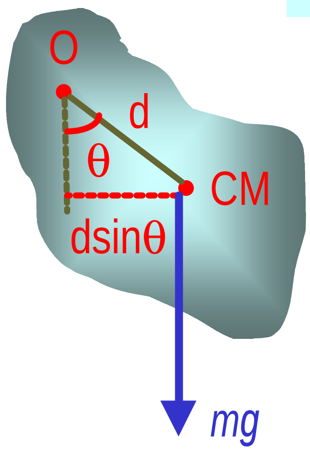
$$L = \frac{T^2 g}{4p^2} = \frac{1 \times 9.8}{4p^2} = 0.248\,m$$

Therefore the difference in length with respect to the current definition of 1m is

$$\Delta L = 1 - L = 1 - 0.248 = 0.752\,m$$

Physical Pendulum

Physical pendulum is an object that oscillates about a fixed axis which does not go through the object's center of mass.



Consider a rigid body pivoted at a point O that is a distance d from the CM.

The magnitude of the net torque provided by the gravity is

$$\sum \tau = -mgd\sin\theta$$

Then

$$\sum \tau = I\alpha = I\frac{d^2\theta}{dt^2} = -mgd\sin\theta$$

Therefore, one can rewrite

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\sin\theta \approx -\left(\frac{mgd}{I}\right)\theta = -\omega^2\theta$$

Thus, the angular frequency ω is

$$\omega = \sqrt{\frac{mgd}{I}}$$

And the period for this motion is

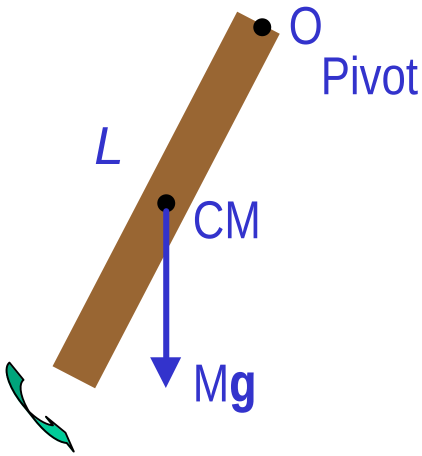
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$$

By measuring the period of physical pendulum, one can measure moment of inertia.

Does this work for simple pendulum?

Example 13.6

A uniform rod of mass M and length L is pivoted about one end and oscillates in a vertical plane. Find the period of oscillation if the amplitude of the motion is small.



Moment of inertia of a uniform rod, rotating about the axis at one end is

$$I = \frac{1}{3}ML^2$$

The distance d from the pivot to the CM is L/2, therefore the period of this physical pendulum is

$$T = \frac{2p}{w} = 2p\sqrt{\frac{I}{Mgd}} = 2p\sqrt{\frac{2ML^2}{3MgL}} = 2p\sqrt{\frac{2L}{3g}}$$

Calculate the period of a meter stick that is pivot about one end and is oscillating in a vertical plane.

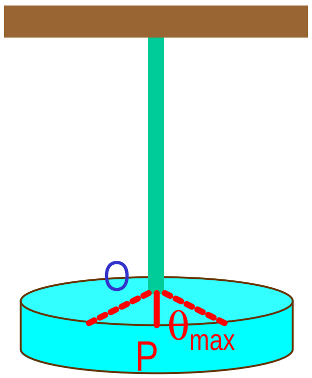
Since L=1m, the period is

$$T = 2p\sqrt{\frac{2L}{3g}} = 2p\sqrt{\frac{2}{3\cdot 9.8}} = 1.64s$$

So the frequency is

$$f = \frac{1}{T} = 0.61s^{-1}$$

Torsional Pendulum



When a rigid body is suspended by a wire to a fixed support at the top and the body is twisted through some small angle θ, the twisted wire can exert a restoring torque on the body that is proportional to the angular displacement.

The torque acting on the body due to the wire is

$$\tau = -\kappa\theta$$

κ is the torsion constant of the wire

Applying the Newton's second law of rotational motion

$$\sum \tau = I\alpha = I\frac{d^2\theta}{dt^2} = -\kappa\theta$$

Then, again the equation becomes

$$\frac{d^2\theta}{dt^2} = -\left(\frac{\kappa}{I}\right)\theta = -\omega^2\theta$$

Thus, the angular frequency ω is

$$\omega = \sqrt{\frac{\kappa}{I}}$$

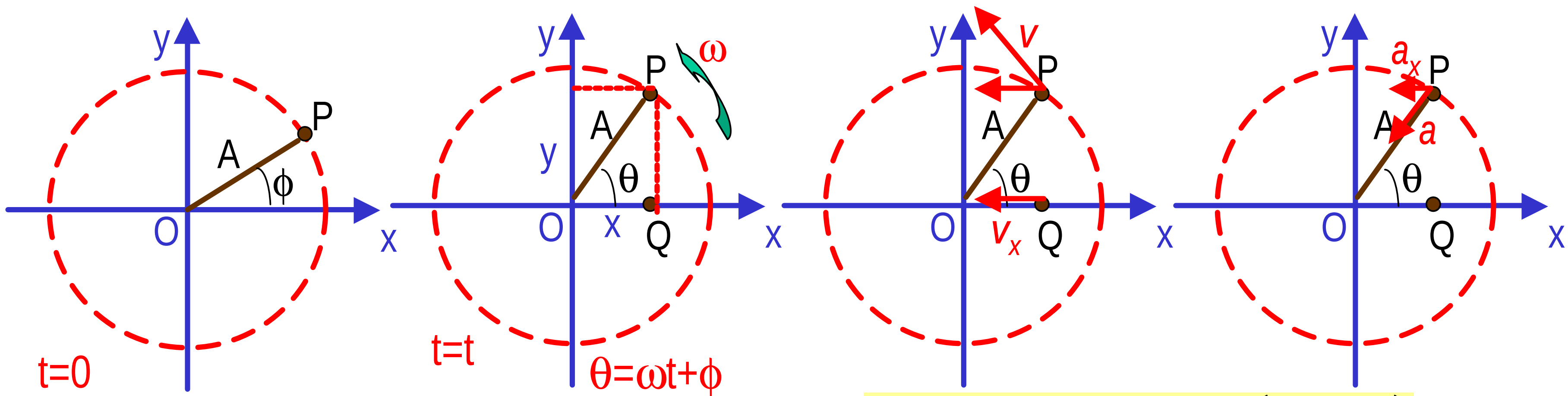
And the period for this motion is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{\kappa}}$$

This result works as long as the elastic limit of the wire is not exceeded

Simple Harmonic and Uniform Circular Motions

Uniform circular motion can be understood as a superposition of two simple harmonic motions in x and y axis.



When the particle rotates at a uniform angular speed ω, x and y coordinate position become

$$x = A\cos\theta = A\cos(\omega t + \phi)$$
$$y = A\sin\theta = A\sin(\omega t + \phi)$$

Since the linear velocity in a uniform circular motion is Aω, the velocity components are

$$v_x = -v\sin\theta = -A\omega\sin(\omega t + \phi)$$
$$v_y = +v\cos\theta = A\omega\cos(\omega t + \phi)$$

Since the radial acceleration in a uniform circular motion is $v^2/A=\omega^2 A$, the components are

$$a_x = -a\cos\theta = -A\omega^2\cos(\omega t + \phi)$$
$$a_y = -a\sin\theta = -A\omega^2\sin(\omega t + \phi)$$

Example 13.7

A particle rotates counterclockwise in a circle of radius 3.00m with a constant angular speed of 8.00 rad/s. At t=0, the particle has an x coordinate of 2.00m and is moving to the right. A) Determine the x coordinate as a function of time.

Since the radius is 3.00m, the amplitude of oscillation in x direction is 3.00m. And the angular frequency is 8.00rad/s. Therefore the equation of motion in x direction is

$$x = A\cos q = (3.00m)\cos(8.00t + f)$$

Since x=2.00, when t=0

$$2.00 = (3.00m)\cos f; \quad f = \cos^{-1}\left(\frac{2.00}{3.00}\right) = 48.2^\circ$$

However, since the particle was moving to the right ϕ=-48.2°,

$$x = (3.00m)\cos(8.00t - 48.2^\circ)$$

Find the x components of the particle's velocity and acceleration at any time t.

Using the displcement

$$v_x = \frac{dx}{dt} = -(3.00 \cdot 8.00)\sin(8.00t - 48.2) = (-24.0m/s)\sin(8.00t - 48.2^\circ)$$

Likewise, from velocity

$$a_x = \frac{dv}{dt} = (-24.0 \cdot 8.00)\cos(8.00t - 48.2) = (-192m/s^2)\cos(8.00t - 48.2^\circ)$$

Damped Oscillation

More realistic oscillation where an oscillating object loses its mechanical energy in time by a retarding force such as friction or air resistance.

Let's consider a system whose retarding force is air resistance R=-b**v** (b is called damping coefficient) and restoration force is -kx

$$\sum F_x = -kx - bv = ma_x$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

The solution for the above 2nd order differential equation is

$$x = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi)$$

The angular frequency ω for this motion is

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

This equation of motion tells us that when the retarding force is much smaller than restoration force, the system oscillates but the amplitude decreases, and ultimately, the oscillation stops.

We express the angular frequency as

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

Where the natural frequency ω_0

$$\omega_0 = \sqrt{\frac{k}{m}}$$

More on Damped Oscillation

The motion is called **Underdamped** when the magnitude of the maximum retarding force $R_{max} = bv_{max} < kA$

How do you think the damping motion would change as retarding force changes?

$$-bv_{\max} \rightarrow -kA$$

As the retarding force become larger, the amplitude reduces more rapidly, eventually stopping its equilibrium position

Under what condition this system does not oscillate?

$$w = 0$$

$$w_0 = \frac{b}{2m}$$

$$b = 2mw_0 = 2\sqrt{mk}$$

The system is **Critically damped**

What do you think happen?

Once released from non-equilibrium position, the object would return to its equilibrium position and stops.

If the retarding force is larger than restoration force

$$R_{\max} = bv_{\max} > kA$$

The system is **Overdamped**

Once released from non-equilibrium position, the object would return to its equilibrium position and stops, but a lot slower than before

Rotational Kinematics

The first type of motion we have learned in linear kinematics was under a constant acceleration. We will learn about the rotational motion under constant acceleration, because these are the simplest motions in both cases.

Just like the case in linear motion, one can obtain

Angular Speed under constant angular acceleration:

$$\omega_f = \omega_i + \alpha t$$

Angular displacement under constant angular acceleration:

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$$

One can also obtain

$$\omega_f^2 = \omega_i^2 + 2\alpha\left(\theta_f - \theta_i\right)$$

Example 10.1

A wheel rotates with a constant angular acceleration pf 3.50 rad/s^2. If the angular speed of the wheel is 2.00 rad/s at t_i=0, a) through what angle does the wheel rotate in 2.00s?

Using the angular displacement formula in the previous slide, one gets

$$\theta_f - \theta_i = \omega t + \frac{1}{2}\alpha t^2$$
$$= 2.00 \times 2.00 + \frac{1}{2} 3.50 \times (2.00)^2$$
$$= 11.0\,rad \quad = \frac{11.0}{2\pi} rev. = 1.75\,rev.$$

What is the angular speed at t=2.00s?

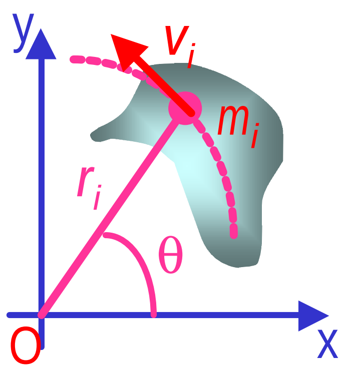
Using the angular speed and acceleration relationship

$$\omega_f = \omega_i + \alpha t$$
$$= 2.00 + 3.50 \times 2.00$$
$$= 9.00\,rad/s$$

Find the angle through which the wheel rotates between t=2.00 s and t=3.00 s.

$$\theta_2 = 11.0\,rad$$
$$\theta_3 = 2.00 \times 3.00 + \frac{1}{2} 3.50 \times (3.00)^2 = 21.8\,rad$$
$$\Delta\theta = \theta_3 - \theta_2 = 10.8\,rad \quad = \frac{10.8}{2\pi} rev. = 1.72\,rev.$$

Rotational Energy



What do you think the kinetic energy of a rigid object that is undergoing a circular motion is?

Kinetic energy of a masslet, m_i, moving at a tangential speed, v_i, is

$$K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 \omega^2$$

Since a rigid body is a collection of masslets, the total kinetic energy of the rigid object is

$$K_R = \sum_i K_i = \frac{1}{2} \sum_i m_i r_i^2 \omega^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

By defining a new quantity called, Moment of Inertia, I, as

$$I = \sum_i m_i r_i^2$$

The above expression is simplified as

$$K_R = \frac{1}{2} I \omega^2$$

What are the dimension and unit of Moment of Inertia? $kg \cdot m^2$ $\left[ML^2\right]$

What do you think the moment of inertia is?

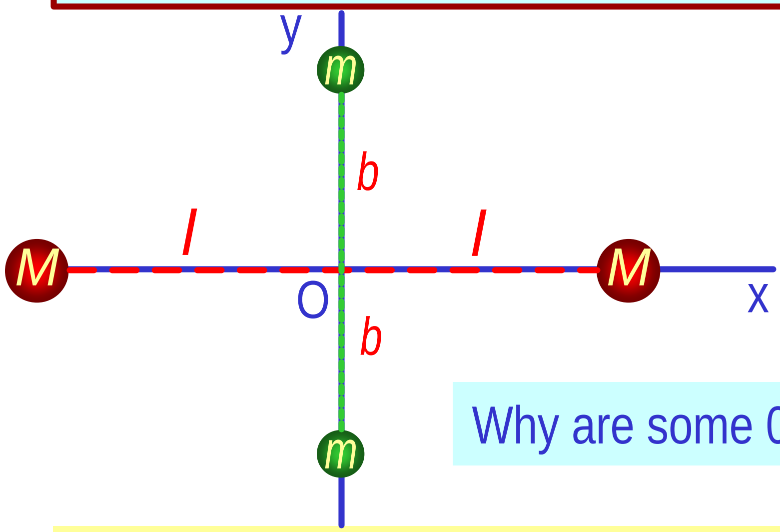
Measure of resistance of an object to changes in its rotational motion.

What similarity do you see between rotational and linear kinetic energies?

Mass and speed in linear kinetic energy are replaced by moment of inertia and angular speed.

Example 10.4

In a system consists of four small spheres as shown in the figure, assuming the radii are negligible and the rods connecting the particles are massless, compute the moment of inertia and the rotational kinetic energy when the system rotates about the y-axis at ω.



Since the rotation is about y axis, the moment of inertia about y axis, I_y, is

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + m \cdot 0^2 + m \cdot 0^2 = 2Ml^2$$

Why are some 0s?

This is because the rotation is done about y axis, and the radii of the spheres are negligible.

Thus, the rotational kinetic energy is

$$K_R = \frac{1}{2} I w^2 = \frac{1}{2}\left(2Ml^2\right)w^2 = Ml^2 w^2$$

Find the moment of inertia and rotational kinetic energy when the system rotates on the x-y plane about the z-axis that goes through the origin O.

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + mb^2 + mb^2 = 2\left(Ml^2 + mb^2\right)$$

$$K_R = \frac{1}{2} I w^2 = \frac{1}{2}\left(2Ml^2 + 2mb^2\right)w^2 = \left(Ml^2 + mb^2\right)w^2$$

Calculation of Moments of Inertia

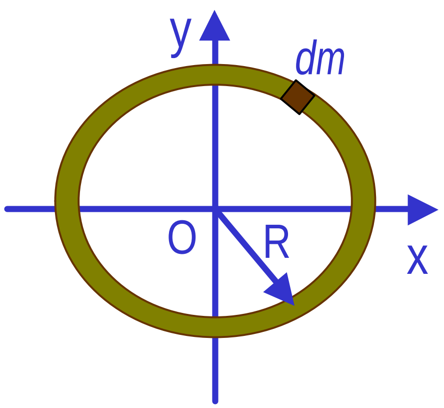
Moments of inertia for large objects can be computed, if we assume the object consists of small volume elements with mass, Δm_i.

The moment of inertia for the large rigid object $I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$

It is sometimes easier to compute moments of inertia in terms of volume of the elements rather than their mass

Using the volume density, ρ, replace d*m* in the above equation with dV. $\rho = \frac{dm}{dV}; \quad dm = \rho dV$ The moments of inertia becomes $I = \int \rho r^2 dV$

Example 10.5: Find the moment of inertia of a uniform hoop of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center.



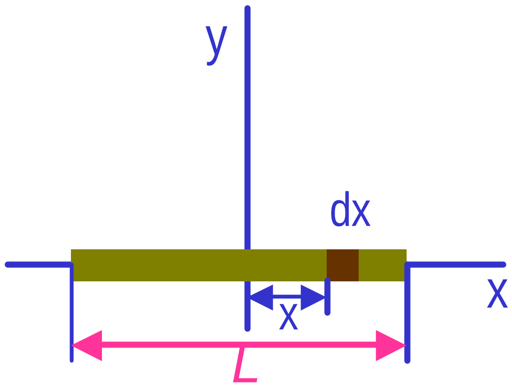
The moment of inertia is $I = \int r^2 dm = R^2 \int dm = MR^2$

What do you notice from this result?

The moment of inertia for this object is the same as that of a point of mass M at the distance R.

Example 10.6

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod and passing through its center of mass.



The line density of the rod is $\lambda = \frac{M}{L}$

so the masslet is $dm = \lambda dx = \frac{M}{L} dx$

The moment of inertia is

$$I = \int r^2 dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

What is the moment of inertia when the rotational axis is at one end of the rod.

$$I = \int r^2 dm = \int_0^L \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_0^L$$

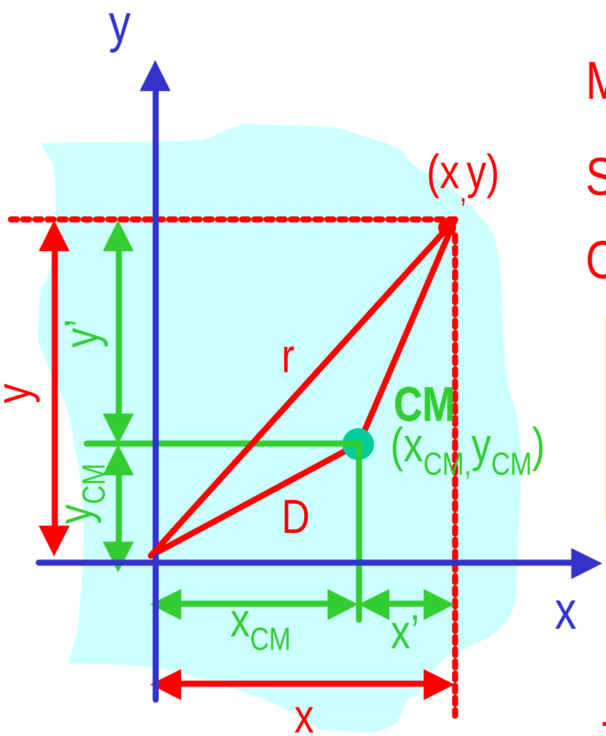
$$= \frac{M}{3L}\left[(L)^3 - 0\right] = \frac{M}{3L}\left(L^3\right) = \frac{ML^2}{3}$$

Will this be the same as the above. Why or why not?

Since the moment of inertia is resistance to motion, it makes perfect sense for it to be harder to move when it is rotating about the axis at one end.

Parallel Axis Theorem

Moments of inertia for highly symmetric object is relatively easy if the rotational axis is the same as the axis of symmetry. However if the axis of rotation does not coincide with axis of symmetry, the calculation can still be done in simple manner using **parallel-axis theorem**. $I = I_{CM} + MD^2$



Moment of inertia is defined $I = \int r^2 dm = \int \sqrt{\left(x^2 + y^2\right)}\, dm \quad (1)$

Since x and y are $x = x_{CM} + x'; \quad y = y_{CM} + y'$

One can substitute x and y in Eq. 1 to obtain

$$I = \int \left[\left(x_{CM} + x'\right)^2 + \left(y_{CM} + y'\right)^2 \right] dm$$

$$= \left(x_{CM}^2 + y_{CM}^2\right) \int dm + 2x_{CM} \int x' dm + 2y_{CM} \int y' dm + \int \left(x'^2 + y'^2\right) dm$$

Since the x' and y' are the distance from CM, by definition

$$\int x' dm = 0$$

$$\int y' dm = 0$$

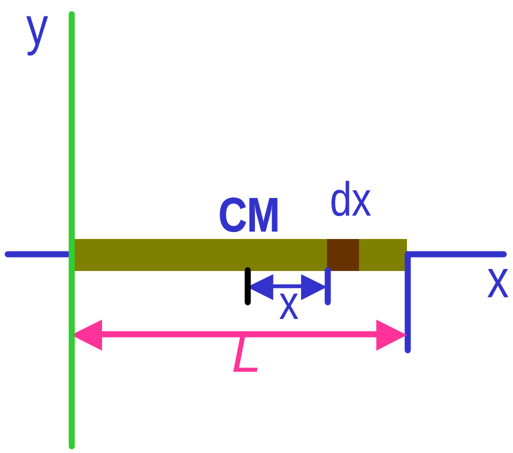
Therefore, the parallel-axis theorem $I = I_{CM} + MD^2$

What does this theorem tell you?

Moment of inertia of any object about any arbitrary axis are the same as the sum of moment of inertia for a rotation about the CM and that of the CM about the rotation axis.

Example 10.8

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis that goes through one end of the rod, using parallel-axis theorem.



The line density of the rod is

$$l = \frac{M}{L}$$

so the masslet is

$$dm = l\,dx = \frac{M}{L}dx$$

The moment of inertia about the CM

$$I_{CM} = \int r^2 dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

Using the parallel axis theorem

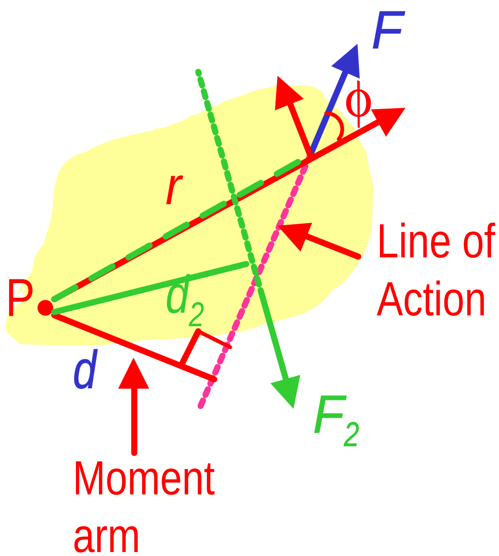
$$I = I_{CM} + D^2 M = \frac{ML^2}{12} + \left(\frac{L}{2}\right)^2 M = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}$$

The result is the same as using the definition of moment of inertia.

Parallel-axis theorem is useful to compute moment of inertia of a rotation of a rigid object with complicated shape about an arbitrary axis

Torque

Torque is the tendency of a force to rotate an object about some axis. Torque, t, is a vector quantity.



Consider an object pivoting about the point P by the force *F* being exerted at a distance r.

The line that extends out of the tail of the force vector is called the line of action.

The perpendicular distance from the pivoting point P to the line of action is called Moment arm.

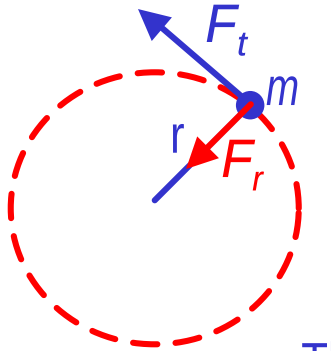
Magnitude of torque is defined as the product of the force exerted on the object to rotate it and the moment arm.

$$t \equiv rF\sin f = Fd$$

When there are more than one force being exerted on certain points of the object, one can sum up the torque generated by each force vectorially. The convention for sign of the torque is positive if rotation is in counter-clockwise and negative if clockwise.

$$\sum t = t_1 + t_2$$
$$= Fd - F_2 d_2$$

Torque & Angular Acceleration



Let's consider a point object with mass *m* rotating in a circle.

What forces do you see in this motion?

The tangential force F_t and radial force F_r

The tangential force F_t is $F_t = ma_t = mra$

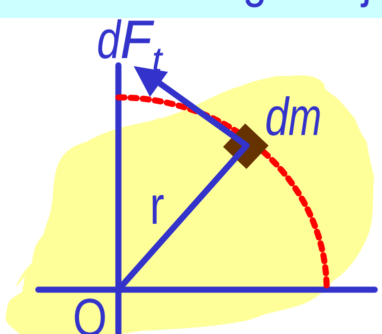
The torque due to tangential force F_t is $t = F_t r = ma_t r = mr^2 a$

What do you see from the above relationship? $t = Ia$

What does this mean? Torque acting on a particle is proportional to the angular acceleration.

What law do you see from this relationship? Analogs to Newton's 2nd law of motion in rotation.

How about a rigid object?



The external tangential force dF_t is $dF_t = dma_t = dmra$

The torque due to tangential force F_t is $dt = dF_t r = (r^2 dm)a$

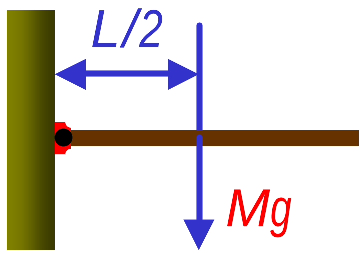
The total torque is $\sum t = a\int r^2 dm = Ia$

What is the contribution due to radial force and why?

Contribution from radial force is 0, because its line of action passes through the pivoting point, making the moment arm 0.

Example 10.10

A uniform rod of length *L* and mass *M* is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane. The rod is released from rest in the horizontal position what is the initial angular acceleration of the rod and the initial linear acceleration of its right end?



The only force generating torque is the gravitational force *Mg*

$$t = Fd = F\frac{L}{2} = Mg\frac{L}{2} = Ia$$

Since the moment of inertia of the rod when it rotates about one end is

$$I = \int_0^L r^2 dm = \int_0^L x^2 l dx = \left(\frac{M}{L}\right)\left[\frac{x^3}{3}\right]_0^L = \frac{ML^2}{3}$$

We obtain

$$a = \frac{MgL}{2I} = \frac{MgL}{\frac{2ML^2}{3}} = \frac{3g}{2L}$$

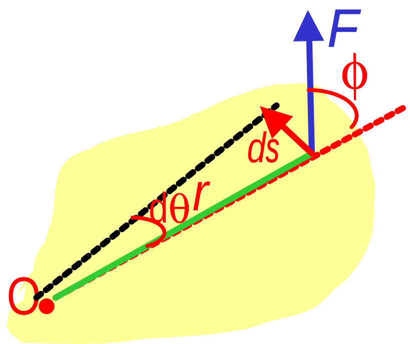
Using the relationship between tangential and angular acceleration

$$a_t = La = \frac{3g}{2}$$

What does this mean?

The tip of the rod falls faster than an object undergoing a free fall.

Work, Power, and Energy in Rotation



Let's consider a motion of a rigid body with a single external force *F* exerted on the point P, moving the object by *ds*.

The work done by the force *F* as the object rotates through infinitesimal distance ds=rdθ in a time dt is

$$dW = \vec{F} \cdot d\vec{s} = (F \sin f)rdq$$

What is *F*sinφ? The tangential component of force *F*.

What is the work done by radial component *F*cosφ? Zero, because it is perpendicular to the displacement.

Since the magnitude of torque is r*F*sinφ,

$$dW = t dq$$

The rate of work, or power becomes

$$P = \frac{dW}{dt} = \frac{t dq}{dt} = t w$$

How was the power defined in linear motion?

The rotational work done by an external force equals the change in rotational energy.

$$\sum t = Ia = I\left(\frac{dw}{dt}\right) = I\left(\frac{dw}{dq}\right)\left(\frac{dq}{dt}\right)$$

The work put in by the external force then

$$dW = \sum t\, dq = Iwdw$$

$$\sum W = \int_{q_i}^{q_i} \sum t\, dq = \int_{w_i}^{w_f} Iwdw = \frac{1}{2} I w_i^2 - \frac{1}{2} I w_f^2$$

Similarity Between Linear and Rotational Motions

All physical quantities in linear and rotational motions show striking similarity.

Similar Quantity	Linear	Rotational
Mass	Mass M	Moment of Inertia $I = \int r^2 dm$
Length of motion	Distance L	Angle q (Radian)
Speed	$v = \frac{dr}{dt}$	$w = \frac{dq}{dt}$
Acceleration	$a = \frac{dv}{dt}$	$a = \frac{dw}{dt}$
Force	Force $F = ma$	Torque $t = Ia$
Work	Work $W = \int_{x_i}^{x_f} F dx$	Work $W = \int_{q_i}^{q_f} t\, dq$
Power	$P = \vec{F} \cdot \vec{v}$	$P = tw$
Momentum	$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{w}$
Kinetic Energy	Kinetic $K = \frac{1}{2} mv^2$	Rotational $K_R = \frac{1}{2} I w^2$

Rolling Motion of a Rigid Body

What is a rolling motion?

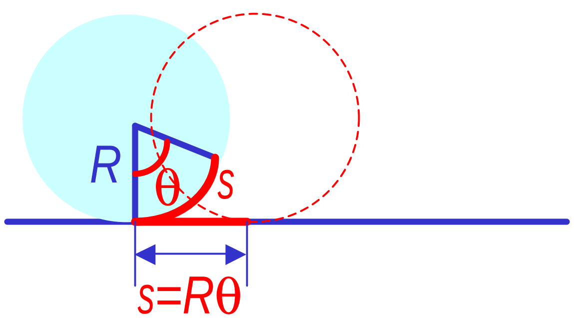
A more generalized case of a motion where the rotational axis moves together with the object

A rotational motion about the moving axis

To simplify the discussion, let's make a few assumptions

1. Limit our discussion on very symmetric objects, such as cylinders, spheres, etc
2. The object rolls on a flat surface

Let's consider a cylinder rolling without slipping on a flat surface



Under what condition does this "Pure Rolling" happen?

The total linear distance the CM of the cylinder moved is

$$s = R\theta$$

Thus the linear speed of the CM is

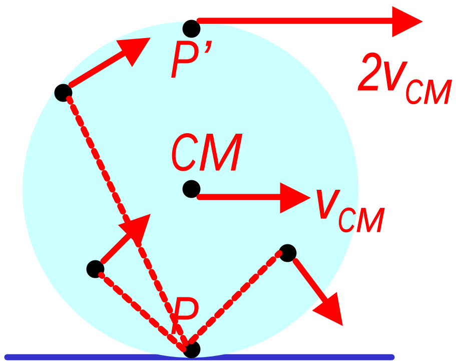
$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

Condition for "Pure Rolling"

Total Kinetic Energy of a Rolling Body

What do you think the total kinetic energy of the rolling cylinder is?

Since it is a rotational motion about the point P, we can writ the total kinetic energy



$$K = \frac{1}{2} I_P w^2$$

Where, I_P, is the moment of inertia about the point P.

Using the parallel axis theorem, we can rewrite

$$K = \frac{1}{2} I_P w^2 = \frac{1}{2} I_{CM} w^2 + \frac{1}{2} MR^2 w^2$$

Since $v_{CM}=R\omega$, the above relationship can be rewritten as

$$K = \frac{1}{2} I_{CM} w^2 + \frac{1}{2} M v_{CM}^2$$

Rotational kinetic energy about the CM

Translational Kinetic energy of the CM

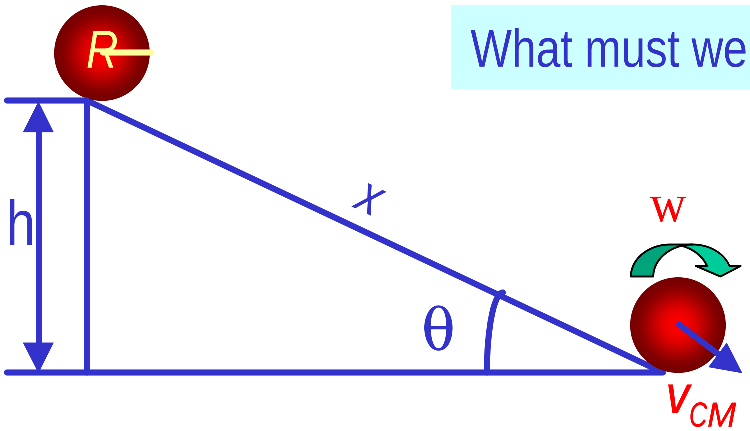
What does this equation mean?

Total kinetic energy of a rolling motion is the sum of the rotational kinetic energy about the CM

And the translational kinetic of the CM

Example 11.1

For solid sphere as shown in the figure, calculate the linear speed of the CM at the bottom of the hill and the magnitude of linear acceleration of the CM.



What must we know first?

The moment of inertia the sphere with respect to the CM!!

$$I_{CM} = \int r^2 dm = \frac{2}{5} MR^2$$

Thus using the formula in the previous slide

$$v_{CM} = \sqrt{\frac{2gh}{1 + I_{CM}/MR^2}} = \sqrt{\frac{2gh}{1+2/5}} = \sqrt{\frac{10}{7}gh}$$

Since h=xsinθ, one obtains

$$v_{CM}^2 = \frac{10}{7} gx \sin\theta$$

Using kinematic relationship

$$v_{CM}^2 = 2a_{CM}x$$

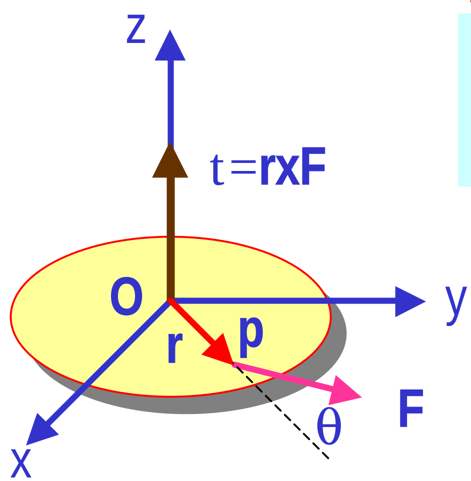
The linear acceleration of the CM is

$$a_{CM} = \frac{v_{CM}^2}{2x} = \frac{5}{7} g \sin\theta$$

What do you see?

Linear acceleration of a sphere does not depend on anything but g and θ.

Torque and Vector Product



Let's consider a disk fixed onto the origin O and the force *F* exerts on the point p. What happens?

The disk will start rotating counter clockwise about the Z axis

The magnitude of torque given to the disk by the force *F* is

$$t = Fr \sin f$$

But torque is a vector quantity, what is the direction?
How is torque expressed mathematically?

$$\boxed{\vec{t} \equiv \vec{r} \times \vec{F}}$$

What is the direction?

The direction of the torque follows the right-hand rule!!

The above quantity is called
Vector product or Cross product

$$\vec{C} \equiv \vec{A} \times \vec{B}$$

$$\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right| \sin q$$

What is the result of a vector product?

Another vector

What is another vector operation we've learned?

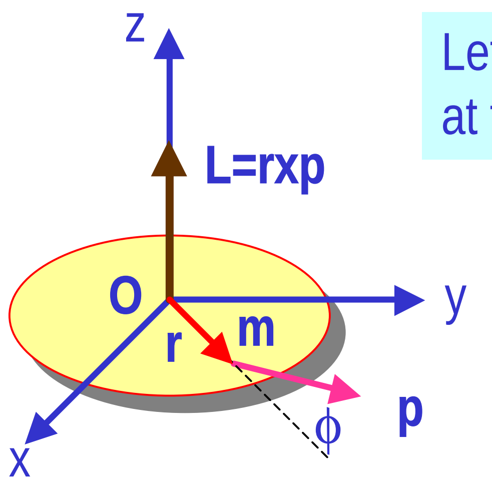
Scalar product

$$C \equiv \vec{A} \cdot \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right| \cos q$$

Result? A scalar

Angular Momentum of a Particle

If you grab onto a pole while running, your body will rotate about the pole, gaining angular momentum. We've used linear momentum to solve physical problems with linear motions, angular momentum will do the same for rotational motions.



Let's consider a point-like object (particle) with mass *m* located at the vector location ***r*** and moving with linear velocity ***v***

The instantaneous angular momentum ***L*** of this particle relative to origin O is

$$\boxed{\vec{L} \equiv \vec{r} \times \vec{p}}$$

What is the unit and dimension of angular momentum?

$kg \cdot m^2 / s^2$

Note that ***L*** depends on origin O.

Why?

Because ***r*** changes

What else do you learn?

The direction of ***L*** is +z

Since ***p*** is *m**v***, the magnitude of ***L*** becomes

$$L = mvr \sin f$$

What do you learn from this?

If the direction of linear velocity points to the origin of rotation, the particle does not have any angular momentum.

If the linear velocity is perpendicular to position vector, the particle moves exactly the same way as a point on a rim.

The point O has to be inertial.

Angular Momentum and Torque

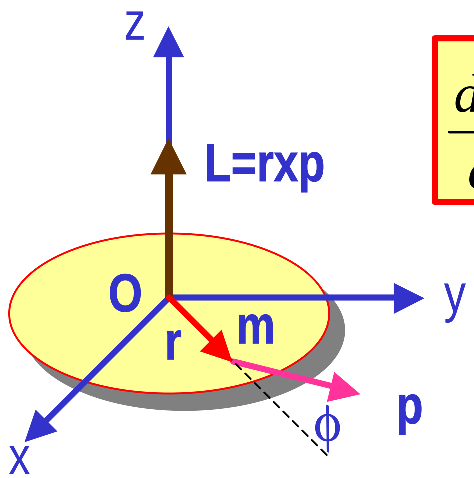
Can you remember how net force exerting on a particle and the change of its linear momentum are related?

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

Total external forces exerting on a particle is the same as the change of its linear momentum.

The same analogy works in rotational motion between torque and angular momentum.

Net torque acting on a particle is $\sum \vec{t} = \vec{r} \times \sum \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}$



$$\frac{d\vec{L}}{dt} = \frac{d\left(\vec{r} \times \vec{p}\right)}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = 0 + \vec{r} \times \frac{d\vec{p}}{dt}$$

Why does this work?

Because *v* is parallel to the linear momentum

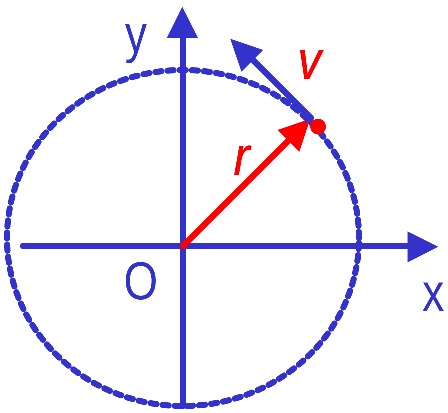
Thus the torque-angular momentum relationship

$$\sum \vec{t} = \frac{d\vec{L}}{dt}$$

The net torque acting on a particle is the same as the time rate change of its angular momentum

Example 11.4

A particle of mass *m* is moving in the xy plane in a circular path of radius r and linear velocity v about the origin O. Find the magnitude and direction of angular momentum with respect to O.



Using the definition of angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m\vec{r} \times \vec{v}$$

Since both the vectors, r and v, are on x-y plane and using right-hand rule, the direction of the angular momentum vector is +z (coming out of the screen)

The magnitude of the angular momentum is $\left|\vec{L}\right| = \left|m\vec{r} \times \vec{v}\right| = mrv \sin f = mrv \sin 90^\circ = mrv$

So the angular momentum vector can be expressed as $\vec{L} = mrv\vec{k}$

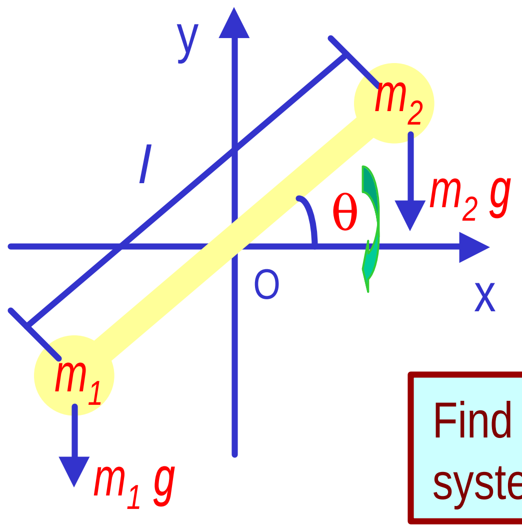
Find the angular momentum in terms of angular velocity w.

Using the relationship between linear and angular speed

$$\vec{L} = mrv\vec{k} = mr^2 w\vec{k} = mr^2\vec{w} = I\vec{w}$$

Example 11.6

A rigid rod of mass *M* and length *l* pivoted without friction at its center. Two particles of mass m_1 and m_2 are connected to its ends. The combination rotates in a vertical plane with an angular speed of ω. Find an expression for the magnitude of the angular momentum.



The moment of inertia of this system is

$$I = I_{rod} + I_{m_1} + I_{m_2} = \frac{1}{12}Ml^2 + \frac{1}{4}m_1 l^2 + \frac{1}{4}m_2 l^2$$

$$= \frac{l^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)$$

$$\boxed{L = Iw = \frac{wl^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)}$$

Find an expression for the magnitude of the angular acceleration of the system when the rod makes an angle θ with the horizon.

If $m_1 = m_2$, no angular momentum because net torque is 0.

If θ=+/–π/2, at equilibrium so no angular momentum.

First compute net external torque

$$t_1 = m_1 g \frac{l}{2}\cos q;\ t_2 = -m_2 g \frac{l}{2}\cos q$$

$$t_{ext} = t_1 + t_2 = \frac{gl\cos q\,(m_1 - m_2)}{2}$$

Thus α becomes

$$a = \frac{\sum t_{ext}}{I} = \frac{\frac{1}{2}(m_1 - m_1)gl\ \cos q}{\frac{l^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)} = \frac{2(m_1 - m_1)\cos q}{\left(\frac{1}{3}M + m_1 + m_2\right)} g/l$$

Example 11.8

A start rotates with a period of 30days about an axis through its center. After the star undergoes a supernova explosion, the stellar core, which had a radius of $1.0\text{x}10^4$km, collapses into a neutron start of radius 3.0km. Determine the period of rotation of the neutron star.

What is your guess about the answer?

The period will be significantly shorter, because its radius got smaller.

Let's make some assumptions:

1. There is no torque acting on it
2. The shape remains spherical
3. Its mass remains constant

Using angular momentum conservation

$$L_i = L_f$$
$$I_i w_i = I_f w_f$$

The angular speed of the star with the period T is

$$w = \frac{2p}{T}$$

Thus

$$w_f = \frac{I_i w_i}{I_f} = \frac{mr_i^2}{mr_f^2}\frac{2p}{T_i}$$

$$T_f = \frac{2p}{w_f} = \left(\frac{r_f^2}{r_i^2}\right)T_i = \left(\frac{3.0}{1.0\times 10^4}\right)^2 \times 30\,days \;= 2.7\times 10^{-6}\,days \;= 0.23\,s$$

Conditions for Equilibrium

What do you think does the term "An object is at its equilibrium" mean?

The object is either at rest (<u>Static Equilibrium</u>) or its center of mass is moving with a constant velocity (Dynamic Equilibrium).

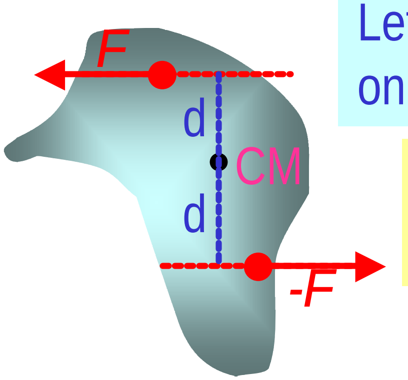
When do you think an object is at its equilibrium?

Translational Equilibrium: Equilibrium in linear motion

$$\sum \vec{F} = 0$$

Is this it?

The above condition is sufficient for a point-like particle to be at its static equilibrium. However for object with size this is not sufficient. One more condition is needed. What is it?



Let's consider two forces equal magnitude but opposite direction acting on a rigid object as shown in the figure. What do you think will happen?

The object will rotate about the CM. The net torque acting on the object about any axis must be 0.

$$\sum \vec{t} = 0$$

For an object to be at its static equilibrium, the object should not have linear or angular speed.

$$v_{CM} = 0 \qquad w = 0$$

More on Conditions for Equilibrium

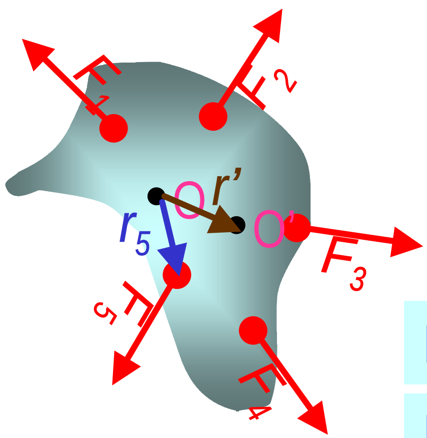
To simplify the problems, we will only deal with forces acting on x-y plane, giving torque only along z-axis. What do you think the conditions for equilibrium be in this case?

The six possible equations from the two vector equations turns to three equations.

$$\sum \vec{F} = 0 \qquad \sum F_x = 0 \qquad \sum F_y = 0$$

$$\sum \vec{t} = 0 \qquad \sum t_z = 0$$

What happens if there are many forces exerting on the object?



If an object is at its translational static equilibrium, and if the net torque acting on the object is 0 about one axis, the net torque must be 0 about any arbitrary axis.

Net Force exerting on the object

$$\sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots = 0$$

Net torque about O

$$\sum \vec{t}_O = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{r}_3 \times \vec{F}_3 + \cdots = \sum \vec{r}_i \times \vec{F}_i = 0$$

Position of force F_i about O'

$$\vec{r}_i' = \vec{r}_i - \vec{r}'$$

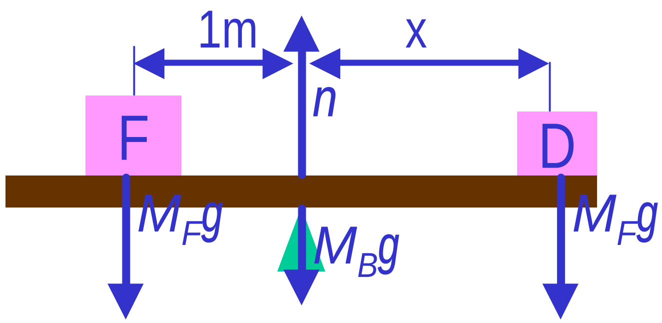
Net torque about O'

$$\sum \vec{t}_{O'} = \vec{r}'_1 \times \vec{F}_1 + \vec{r}'_2 \times \vec{F}_2 + \cdots = \left(\vec{r}_1 - \vec{r}'\right) \times \vec{F}_1 + \left(\vec{r}_2 - \vec{r}'\right) \times \vec{F}_2 + \cdots = \sum \vec{r}_i \times \vec{F}_i - \vec{r}' \times \sum \vec{F}_i$$

$$\sum \vec{t}_{O'} = \sum \vec{r}_i \times \vec{F}_i - \vec{r}' \times 0 = \sum \vec{t}_O = 0$$

Example 12.1

A uniform 40.0 N board supports a father and daughter weighing 800 N and 350 N, respectively. If the support (or fulcrum) is under the center of gravity of the board and the father is 1.00 m from CoG, what is the magnitude of normal force *n* exerted on the board by the support?



Since there is no linear motion, this system is in its translational equilibrium

$$\sum F_x = 0$$

$$\sum F_y = M_B g + M_F g + M_D g - n = 0$$

Therefore the magnitude of the normal force

$$n = 40.0 + 800 + 350 = 1190N$$

Determine where the child should sit to balance the system.

The net torque about the fulcrum by the three forces are

$$t = M_B g \cdot 0 + M_F g \cdot 1.00 - M_D g \cdot x = 0$$

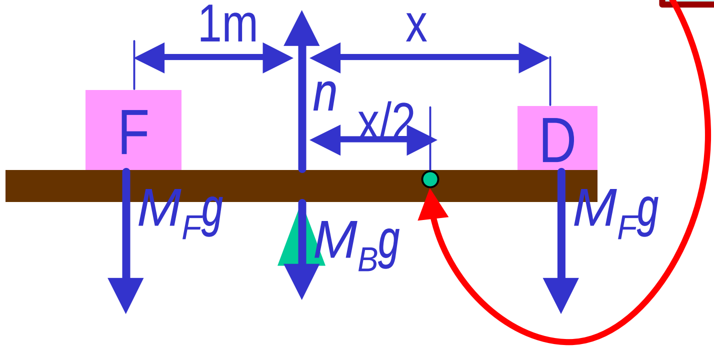
Therefore to balance the system the daughter must sit

$$x = \frac{M_F g}{M_D g} \cdot 1.00m = \frac{800}{350} \cdot 1.00m = 2.29m$$

Example 12.1 Continued

Rotational axis

Determine the position of the child to balance the system for different position of axis of rotation.



The net torque about the axis of rotation by all the forces are

$$t = M_B g \cdot x/2 + M_F g \cdot (1.00 + x/2) - n \cdot x/2 - M_D g \cdot x/2 = 0$$

Since the normal force is $n = M_B g + M_F g + M_D g$

The net torque can be rewritten

$$t = M_B g \cdot x/2 + M_F g \cdot (1.00 + x/2)$$
$$- (M_B g + M_F g + M_D g) \cdot x/2 - M_D g \cdot x/2$$
$$= M_F g \cdot 1.00 - M_D g \cdot x = 0$$

Therefore

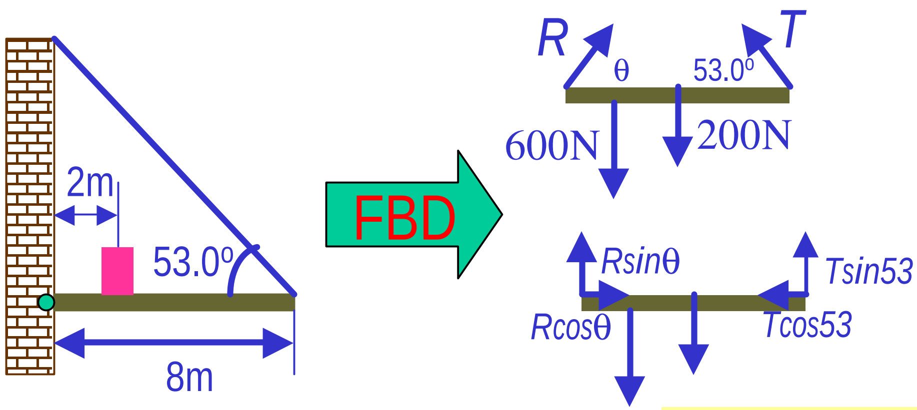
$$x = \frac{M_F g}{M_D g} \cdot 1.00m = \frac{800}{350} \cdot 1.00m = 2.29m$$

What do we learn?

No matter where the rotation axis is, net effect of the torque is identical.

Example 12.3

A uniform horizontal beam with a length of 8.00m and a weight of 200N is attached to a wall by a pin connection. Its far end is supported by a cable that makes an angle of 53.0º with the horizontal. If 600N person stands 2.00m from the wall, find the tension in the cable, as well as the magnitude and direction of the force exerted by the wall on the beam.



First the translational equilibrium, using components

$$\sum F_x = R\cos\theta - T\cos 53.0^\circ = 0$$

$$\sum F_y = R\sin\theta + T\sin 53.0^\circ - 600N - 200N = 0$$

From the rotational equilibrium

$$\sum \tau = T\sin 53.0^\circ \times 8.00 - 600N \times 2.00 - 200N \cdot 4.00m = 0$$

$$T = 313N$$

Using the translational equilibrium

$$R\cos\theta = T\cos 53.0^\circ$$

$$R\sin\theta = -T\sin 53.0^\circ + 600N + 200N$$

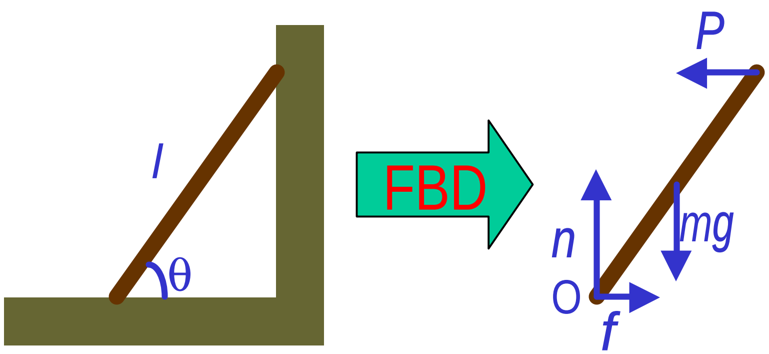
$$\theta = \tan^{-1}\left(\frac{800 - 313\times\sin 53.0^\circ}{313\cos 53.0^\circ}\right) = 71.7^\circ$$

And the magnitude of R is

$$R = \frac{T\cos 53.0^\circ}{\cos\theta} = \frac{313\times\cos 53.0^\circ}{\cos 71.1^\circ} = 582N$$

Example 12.4

A uniform ladder of length *l* and weight *mg*=50 N rests against a smooth, vertical wall. If the coefficient of static friction between the ladder and the ground is μ_s=0.40, find the minimum angle θ_{min} at which the ladder does not slip.



First the translational equilibrium, using components

$$\sum F_x = f - P = 0$$

$$\sum F_y = -mg + n = 0$$

Thus, the normal force is

$$n = mg = 50\,N$$

The maximum static friction force just before slipping is, therefore,

$$f_s^{\max} = \mu_s n = 0.4 \times 50\,N = 20\,N = P$$

From the rotational equilibrium

$$\sum \tau_O = -mg\frac{l}{2}\cos\theta_{\min} + Pl\sin\theta_{\min} = 0$$

$$\theta_{\min} = \tan^{-1}\left(\frac{mg}{2P}\right) = \tan^{-1}\left(\frac{50N}{40N}\right) = 51^{\circ}$$

Example 12.7

A solid brass sphere is initially under normal atmospheric pressure of $1.0\times10^5 N/m^2$. The sphere is lowered into the ocean to a depth at which the pressures is $2.0\times10^7 N/m^2$. The volume of the sphere in air is $0.5m^3$. By how much its volume change once the sphere is submerged?

Since bulk modulus is $$B = -\frac{\Delta P}{\Delta V / V_i}$$

The amount of volume change is $$\Delta V = -\frac{\Delta P V_i}{B}$$

From table 12.1, bulk modulus of brass is 6.1×10^{10} N/m^2

The pressure change ΔP is $$\Delta P = P_f - P_i = 2.0\times10^7 - 1.0\times10^5 \approx 2.0\times10^7$$

Therefore the resulting volume change ΔV is $$\Delta V = V_f - V_i = -\frac{2.0\times10^7\times0.5}{6.1\times10^{10}} = -1.6\times10^{-4} m^3$$

The volume has decreased.