1443-501 Spring 2002
Lecture #21

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1. Kepler's Laws
2. The Law of Gravity & The Motion of Planets
3. The Gravitational Field
4. Gravitational Potential Energy
5. Energy in Planetary and Satellite Motions

Today's Homework Assignment would have been #10 but I will assign next Monday.

Example 14.3

Using the fact that g=9.80m/s^2 at the Earth's surface, find the average density of the Earth.

Since the gravitational acceleration is

$$g = G\frac{M_E}{R_E^{\,2}} = 6.67\times10^{-11}\frac{M_E}{R_E^{\,2}}$$

So the mass of the Earth is

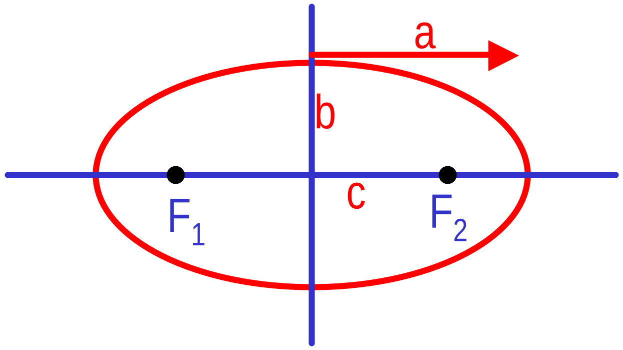
$$M_E = \frac{R_E^{\,2}g}{G}$$

Therefore the density of the Earth is

$$\rho = \frac{M_E}{V_E} = \frac{\frac{R_E^{\,2}g}{G}}{\frac{4\pi}{3}R_E^{\,3}} = \frac{3g}{4\pi G R_E}$$

$$= \frac{3\times9.80}{4\pi\times6.67\times10^{-11}\times6.37\times10^{6}} = 5.50\times10^{3}kg/m^{3}$$

Kepler's Laws & Ellipse



Ellipses have two different axis, major (long) and minor (short) axis, and two focal points, F_1 & F_2

a is the length of a semi-major axis

b is the length of a semi-minor axis

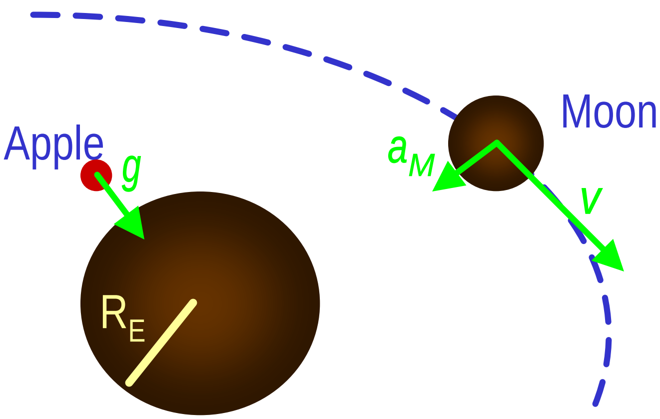
Kepler lived in Germany and discovered the law's governing planets' movement some 70 years before Newton, by analyzing data.

- All planets move in elliptical orbits with the Sun at one focal point.
- The radius vector drawn from the Sun to a planet sweeps out equal area in equal time intervals. *(Angular momentum conservation)*
- The square of the orbital period of any planet is proportional to the cube of the semi-major axis of the elliptical orbit.

Newton's laws explain the cause of the above laws. Kepler's third law is the direct consequence of law of gravitation being inverse square law.

The Law of Gravity and the Motion of Planets

- Newton assumed that the law of gravitation applies the same whether it is on the Moon or the apple on the surface of the Earth.
- The interacting bodies are assumed to be point like particles.



Newton predicted that the ratio of the Moon's acceleration a_M to the apple's acceleration g would be

$$\frac{a_M}{g} = \frac{(1/r_M)^2}{(1/R_E)^2} = \left(\frac{R_E}{r_M}\right)^2 = \left(\frac{6.37\times10^6}{3.84\times10^8}\right)^2 = 2.75\times10^{-4}$$

Therefore the centripetal acceleration of the Moon, a_M, is

$$a_M = 2.75\times10^{-4}\times9.80 = 2.70\times10^{-3}\,m/s^2$$

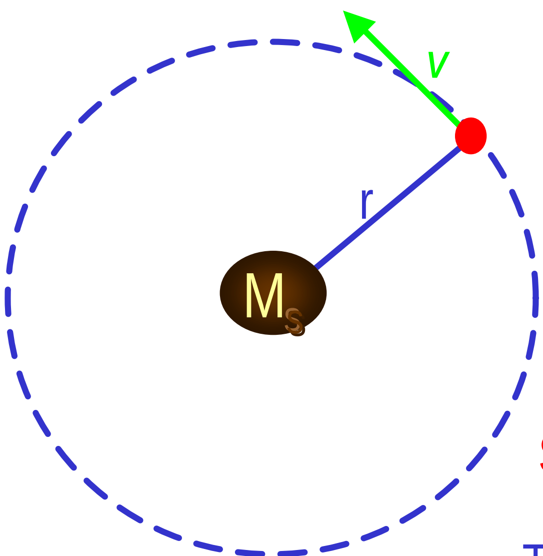
Newton also calculated the Moon's orbital acceleration a_M from the knowledge of its distance from the Earth and its orbital period, T=27.32 days=2.36x10^{6}s

$$a_M = \frac{v^2}{r_M} = \frac{(2pr_M/T)^2}{r_M} = \frac{4p^2 r_M}{T} = \frac{4p^2\times3.84\times10^8}{2.36\times10^6} = 2.72\times10^{-3}\,m/s^2 \approx \frac{9.80}{(60)^2}$$

This means that the Moon's distance is about 60 times that of the Earth's radius, its acceleration is reduced by the square of the ratio. This proves that the inverse square law is valid.

Kepler's Third Law

It is crucial to show that Keper's third law can be predicted from the inverse square law for circular orbits.



Since the gravitational force exerted by the Sun is radially directed toward the Sun to keep the planet circle, we can apply Newton's second law

$$\frac{GM_s M_P}{r^2} = \frac{M_p v^2}{r}$$

Since the orbital speed, v, of the planet with period T is $v = \frac{2\pi r}{T}$

The above can be written $\frac{GM_s M_P}{r^2} = \frac{M_p (2\pi r / T)^2}{r}$

Solving for T one can obtain

$$T = \left(\frac{4\pi^2}{GM_s}\right) r^3 = K_s r^3$$

and

$$K_s = \left(\frac{4\pi^2}{GM_s}\right) = 2.97 \times 10^{-19} s^2 / m^3$$

This is Keper's third law. It's also valid for ellipse for r being the length of the semi-major axis. The constant K_s is independent of mass of the planet.

Example 14.4

Calculate the mass of the Sun using the fact that the period of the Earth's orbit around the Sun is 3.16×10^7s and its distance from the Sun is 1.496×10^{11}m.

Using Kepler's third law.

$$T = \left(\frac{4p^2}{GM_s} \right) r^3 = K_s r^3$$

The mass of the Sun, M_s, is

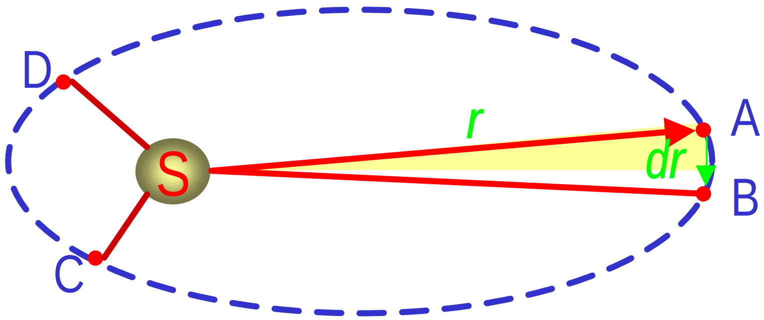
$$M_s = \left(\frac{4p^2}{GT} \right) r^3$$

$$= \left(\frac{4p^2}{6.67 \times 10^{-11} \times 3.16 \times 10^7} \right) \times \left(1.496 \times 10^{11}\right)^3$$

$$= 1.99 \times 10^{30}\, kg$$

Kepler's Second Law and Angular Momentum Conservation

Consider a planet of mass M_p moving around the Sun in an elliptical orbit.



Since the gravitational force acting on the planet is always toward radial direction, it is a *central force*

Therefore the torque acting on the planet by this force is always 0.

$$t = \vec{r} \times \vec{F} = \vec{r} \times F\hat{r} = 0$$

Since torque is the time rate change of angular momentum ***L***, the angular momentum is constant.

$$t = \frac{d\vec{L}}{dt} = 0; \quad \vec{L} = const$$

Because the gravitational force exerted on a planet by the Sun results in no torque, the angular momentum L of the planet is constant.

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times M_p \vec{v} = M_p \vec{r} \times \vec{v} = const$$

Since the area swept by the motion of the planet is

$$dA = \frac{1}{2}\left|\vec{r} \times d\vec{r}\right| = \left|\vec{r} \times \vec{v}dt\right| = \frac{L}{2M_p}dt \quad \Rightarrow \quad \frac{dA}{dt} = \frac{L}{2M_p} = const$$

This is Keper's second law which states that the radius vector from the Sun to a planet sweeps our equal areas in equal time intervals.

The Gravitational Field

The gravitational force is a field force. The force exists every point in the space.

If one were to place a test object of mass m at a any point in the space in the existence of another object of mass M, the test object will fill the gravitational force, $\vec{F}_g = m\vec{g}$, exerted by M

Therefore the gravitational field *g* is defined as $\vec{g} \equiv \frac{\vec{F}_g}{m}$

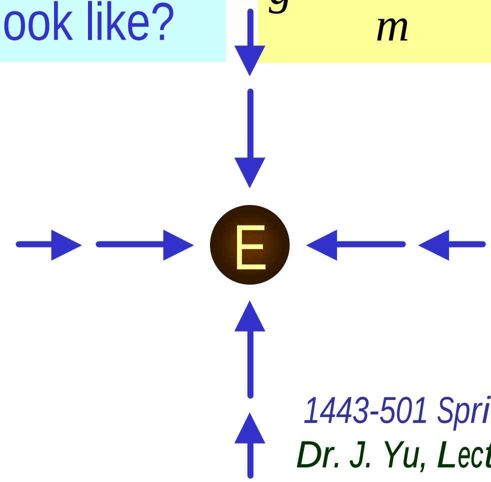
In other words, the gravitational field at a point in space is the gravitational force experienced by a test particle placed at the point divided by the mass of the test particle.

So how does the Earth's gravitational field look like?

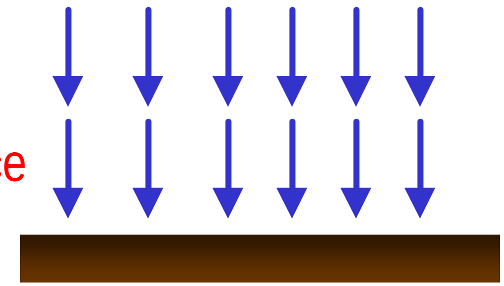
$$\vec{g} = \frac{\vec{F}_g}{m} = -\frac{GM_E}{R_E^2}\hat{r}$$

Where $\hat{r}$ is the unit vector pointing outward from the center of the Earth

Far away from the Earth's surface



Close to the Earth's surface



The Gravitational Potential Energy

What is the potential energy of an object at the height y from the surface of the Earth?

$$U = mgy$$

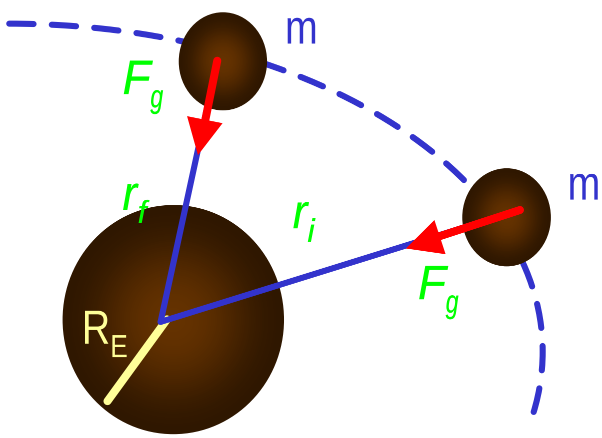
Do you think this would work in general cases?

No, it would not.

Why not?

Because this formula is only valid for the case where the gravitational force is constant, near the surface of the Earth and the generalized gravitational force is inversely proportional to the square of the distance.

OK. Then how would we generalize the potential energy in the gravitational field?



Because gravitational force is a central force and a central force is a conservative force, the work done by the gravitational force is independent of the path.

The path can be looked at as consisting of many tangential and radial motions.

More on The Gravitational Potential Energy

Since the gravitational force is a radial force, it only performed work while the path was radial direction only. Therefore, the work performed by the gravitational force that depends on the position becomes

$$dW = \vec{F} \cdot d\vec{r} = F(r)dr \xrightarrow{\text{For the whole path}} W = \int_{r_i}^{r_f} F(r)dr$$

Therefore the potential energy is the negative change of work in the path

$$\Delta U = U_f - U_i = -\int_{r_i}^{r_f} F(r)dr$$

Since the Earth's gravitational force is

$$F(r) = -\frac{GM_E m}{r^2}$$

So the potential energy function becomes

$$U_f - U_i = \int_{r_i}^{r_f} \frac{GM_E m}{r^2} dr = -GM_E m\left[\frac{1}{r_f} - \frac{1}{r_i}\right]$$

Since potential energy only matters for differences, by taking the infinite distance as the initial point of the potential energy, we get

$$U = -\frac{GM_E m}{r}$$

For any two particles?

$$U = -\frac{Gm_1 m_2}{r}$$

The energy needed to take the particles infinitely apart.

For many particles?

$$U = \sum_{i,j} U_{i,j}$$

Example 14.6

A particle of mass m is displaced through a small vertical distance Δy near the Earth's surface. Show that in this situation the general expression for the change in gravitational potential energy is reduced to the $\Delta U = mg\Delta y$.

Taking the general expression of gravitational potential energy

$$\Delta U = -GM_E m\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

The above formula becomes

$$\Delta U = -GM_E m\frac{(r_f - r_i)}{r_f r_i} = -GM_E m\frac{\Delta y}{r_f r_i}$$

Since the situation is close to the surface of the Earth

$$r_i \approx R_E \text{ and } r_f \approx R_E$$

Therefore, ΔU becomes

$$\Delta U = -GM_E m\frac{\Delta y}{R_E^2}$$

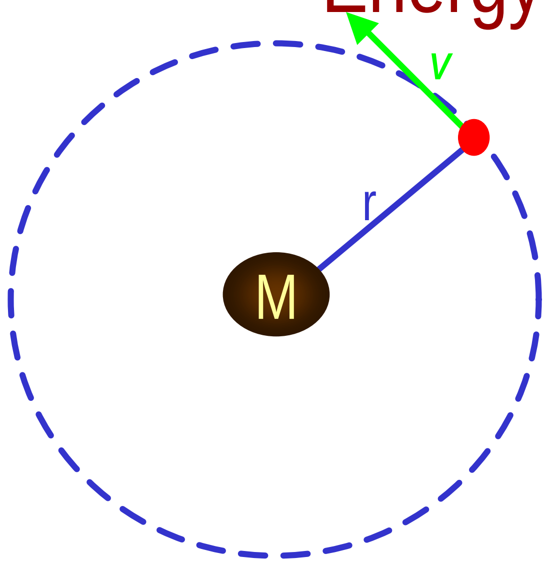
Since on the surface of the Earth the gravitational field is

$$g = \frac{GM_E}{R_E^2}$$

The potential energy becomes

$$\Delta U = -mg\Delta y$$

Energy in Planetary and Satellite Motions



Consider an object of mass m moving at a speed *v* near a massive object of mass M (M>>m).

What's the total energy?

$$E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Systems like Sun and Earth or Earth and Moon whose motions are contained within a closed orbit is called *Bound Systems.*

For a system to be bound, the total energy must be negative.

Assuming a circular orbit, in order for the object to be kept in the orbit the gravitational force must provide the radial acceleration. Therefore from Newton's second law of motion

$$\frac{GM_E m}{r^2} = ma = m\frac{v^2}{r}$$

The kinetic energy for this system is

$$\frac{1}{2}mv^2 = \frac{GM_E m}{2r}$$

Therefore the total mechanical energy of the system is

$$E = K + U = -\frac{GMm}{2r}$$

Since the gravitational force is conservative, the total mechanical energy of the system is conserved.

Example 14.7

The space shuttle releases a 470kg communication satellite while in an orbit that is 280km above the surface of the Earth. A rocket engine on the satellite boosts it into a geosynchronous orbit, which is an orbit in which the satellite stays directly over a single location on the Earth, How much energy did the engine have to provide?

What is the radius of the geosynchronous orbit?

$$T = 1day = 8.64 \times 10^{4} s$$

From Kepler's 3rd law $T^2 = K_E r_{GS}^3$ Where K_E is $K_E = \frac{4p^2}{GM_E} = 9.89 \times 10^{-14} s^2 / m^3$

Therefore the geosynchronous radius is

$$r_{GS} = \sqrt[3]{\frac{T^2}{K_E}} = \sqrt[3]{\frac{\left(8.64 \times 10^4\right)^2}{9.89 \times 10^{-14}}} = \sqrt[3]{\frac{\left(8.64 \times 10^4\right)^2}{9.89 \times 10^{-14}}} = 4.23 \times 10^7 m$$

Because the initial position before the boost is 280km

$$r_i = R_E + 2.80 \times 10^5 m = 6.65 \times 10^6 m$$

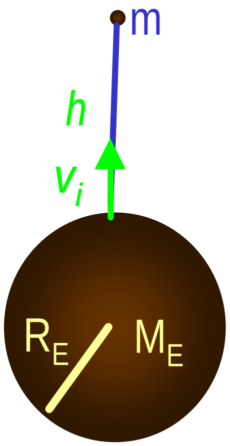
The total energy needed to boost the satellite at the geosynchronous radius is the difference of the total energy before and after the boost

$$\Delta E = -\frac{GM_E m_s}{2}\left(\frac{1}{r_{GS}} - \frac{1}{r_i}\right)$$

$$= -\frac{6.67\times10^{-11}\times5.98\times10^{24}\times470}{2}\left(\frac{1}{4.23\times10^{7}} - \frac{1}{6.65\times10^{6}}\right) = 1.19\times10^{10} J$$

Escape Speed

v_f=0 at $h=r_{max}$



Consider an object of mass m is projected vertically from the surface of the Earth with an initial speed v_i and eventually comes to stop v_f=0 at the distance r_{max}.

Because the total energy is conserved

$$E = K + U = \frac{1}{2}mv_i^2 - \frac{GM_E m}{R_E} = -\frac{GM_E m}{r_{max}}$$

Solving the above equation for v_i, one obtains

$$v_i = \sqrt{2GM_E\left(\frac{1}{R_E} - \frac{1}{r_{max}}\right)}$$

Therefore if the initial speed v_i is known one can use this formula to compute the final height h of the object.

$$h = r_{max} - R_E = \frac{v_i^2 R_E^2}{2GM_E - v_i^2 R_E}$$

In order for the object to escape Earth's gravitational field completely, the initial speed needs to be

$$v_{esc} = \sqrt{\frac{2GM_E}{R_E}} = \sqrt{\frac{2\times 6.67\times 10^{-11}\times 5.98\times 10^{24}}{6.37\times 10^{6}}}$$

$$= 1.12\times 10^{4}\, m/s = 11.2\, km/s$$

This is called the escape speed. This formula is valid for any planet or large mass objects.

How does this depend on the mass of the escaping object?

Independent of the mass of the escaping object