1443-501 Spring 2002
Lecture #24

Dr. Jaehoon Yu

Review of Chap. 1 - 15

Final Exam at 5:30pm, Monday, May 6 (covers Ch 1- 16).

2-dim Motion Under Constant Acceleration

- Position vectors in xy plane: $\vec{r}_i = x_i\vec{i} + y_i\vec{j}$ $\quad \vec{r}_f = x_f\vec{i} + y_f\vec{j}$
- Velocity vectors in xy plane: $\vec{v}_i = v_{xi}\vec{i} + v_{yi}\vec{j}$ $\quad \vec{v}_f = v_{xf}\vec{i} + v_{yf}\vec{j}$

$$v_{xf} = v_{xi} + a_x t, v_{yf} = v_{yi} + a_y t$$
$$\vec{v}_f = (v_{xi} + a_x t)\vec{i} + (v_{yi} + a_y t)\vec{j} = \vec{v}_i + \vec{a}t$$

- How are the position vectors written in acceleration vectors?

$$x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2, y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$$
$$\vec{r}_f = \left(x_i + v_{xi}t + \frac{1}{2}a_x t^2\right)\vec{i} + \left(y_i + v_{yi}t + \frac{1}{2}a_y t^2\right)\vec{j}$$
$$= \vec{r}_i + \vec{v}t + \frac{1}{2}\vec{a}t^2$$

Example 2.12

A stone was thrown straight upward at t=0 with +20.0m/s initial velocity on the roof of a 50.0m high building,

$g=-9.80m/s^2$

1. Find the time the stone reaches at maximum height (*v=0*)
2. Find the maximum height
3. Find the time the stone reaches its original height
4. Find the velocity of the stone when it reaches its original height
5. Find the velocity and position of the stone at t=5.00s

1

$$v_f = v_{yi} + a_y t = +20.0 - 9.80t = 0.00$$

$$t = \frac{20.0}{9.80} = 2.04s$$

2

$$y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$$

$$= 50.0 + 20\times 2.04 + \frac{1}{2}\times(-9.80)\times(2.04)^2$$

$$= 50.0 + 20.4 = 70.4(m)$$

3

$$t = 2.04\times 2 = 4.08s$$

4

$$v_{yf} = v_{yi} + a_y t = 20.0 + (-9.80)\times 4.08 = -20.0(m/s)$$

5-Velocity

$$v_{yf} = v_{yi} + a_y t$$

$$= 20.0 + (-9.80)\times 5.00$$

$$= -29.0(m/s)$$

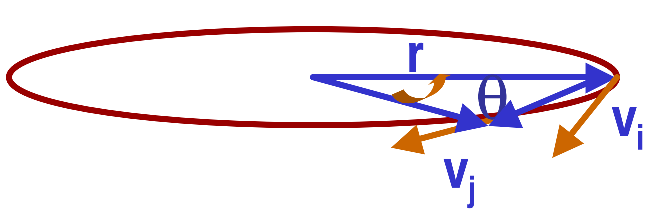
5-Position

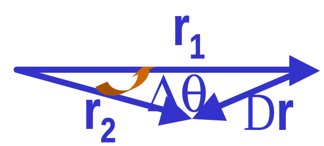
$$y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$$

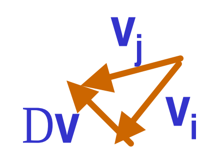
$$= 50.0 + 20.0\times 5.00 + \frac{1}{2}\times(-9.80)\times(5.00)^2 = +27.5(m)$$

Uniform Circular Motion

- A motion with a constant speed on a circular path.
  - The velocity of the object changes, because the direction changes
  - Therefore, there is an acceleration







Angle is Δθ

The acceleration pulls the object inward: *Centripetal Acceleration*

Average Acceleration

$$\vec{a} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} = \frac{\Delta \vec{v}}{\Delta t}$$

$$q = \frac{\left|\Delta \vec{v}\right|}{v} = \frac{\left|\Delta \vec{r}\right|}{r}, \quad \left|\Delta \vec{v}\right| = v\frac{\left|\Delta \vec{r}\right|}{r}, \quad \overline{a} = \frac{v}{\Delta t}\frac{\left|\Delta \vec{r}\right|}{r}$$

Instantaneous Acceleration

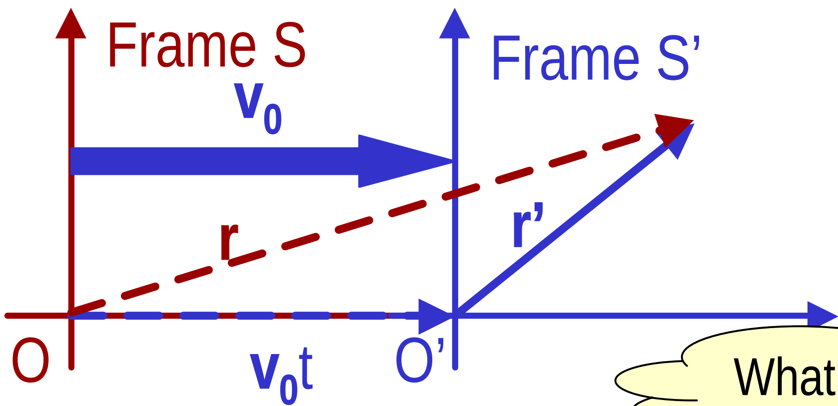
$$a_r = \lim_{\Delta t \to 0} \overline{a} = \lim_{\Delta t \to 0} \frac{\left|\Delta \vec{r}\right|}{\Delta t}\frac{v}{r} = v \times \frac{v}{r} = \frac{v^2}{r}$$

Is this correct in dimension?

What story is this expression telling you?

Relative Velocity and Acceleration

The velocity and acceleration in two different frames of references can be denoted:



Galilean transformation equation

$$\vec{r}\,' = \vec{r} - \vec{v}_0 t$$
$$\frac{d\vec{r}\,'}{dt} = \frac{d\vec{r}}{dt} - \vec{v}_0$$
$$\vec{v}\,' = \vec{v} - \vec{v}_0$$

What does this tell you?

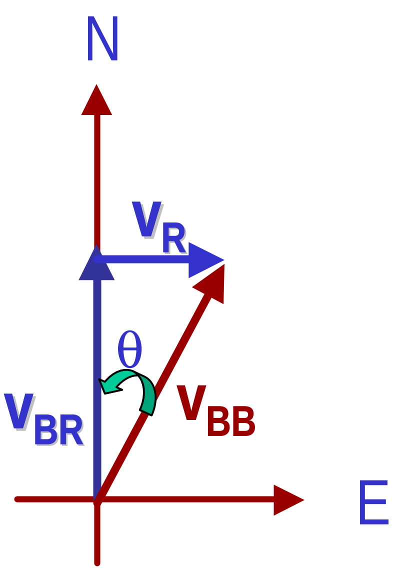
$$\vec{r}\,' = \vec{r} - \vec{v}_0 t$$
$$\frac{d\vec{v}\,'}{dt} = \frac{d\vec{v}}{dt} - \frac{d\vec{v}_0}{dt}$$
$$\vec{a}\,' = \vec{a}, \text{ when } \vec{v}_0 \text{ is constant}$$

The accelerations measured in two frames are the same when the frames move at a constant velocity with respect to each other!!!

The earth's gravitational acceleration is the same in a frame moving at a constant velocity wrt the earth.

Example 4.9

A boat heading due north with a speed 10.0km/h is crossing the river whose stream has a uniform speed of 5.00km/h due east. Determine the velocity of the boat seen by the observer on the bank.



$$\vec{v}_{BB} = \vec{v}_{BR} + \vec{v}_R$$

$$\left|\vec{v}_{BB}\right| = \sqrt{\left|\vec{v}_{BR}\right|^2 + \left|\vec{v}_R\right|^2} = \sqrt{(10.0)^2 + (5.00)^2} = 11.2\,km/h$$

$$\because\ \vec{v}_{BR} = 10.0\,\hat{j} \text{ and } \vec{v}_R = 5.00\,\hat{i}$$

$$\vec{v}_{BB} = 5.00\,\hat{i} + 10.0\,\hat{j}$$

$$\theta = \tan^{-1}\left(\frac{v_{BBy}}{v_{BBx}}\right) = \tan^{-1}\left(\frac{5.00}{10.0}\right) = 26.6^\circ$$

How long would it take for the boat to cross the river if the width is 3.0km?

$$v_{BB}\cos\theta \bullet t = 3.0km$$

$$t = \frac{3.0}{v_{BB}\cos\theta} = \frac{3.0}{11.2\times\cos(26.6^\circ)} = 0.30hrs = 18\min$$

Newton's Laws

1^{st} Law:
Law of Inertia

In the absence of external forces, an object at rest remains at rest and an object in motion continues in motion with a constant velocity.

2^{nd} Law:
Law of Forces

$$\sum_i \vec{F}_i = m\vec{a}$$

The acceleration of an object is directly proportional to the net force exerted on it and inversely proportional to the object's mass.

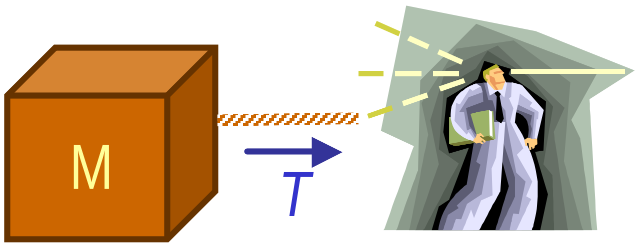
3^{rd} Law:
Law of Action and Reaction

$$\overrightarrow{F_{21}} = -\overrightarrow{F_{12}}$$

If two objects interact, the force, F_{12}, exerted on object 1 by object 2 is equal magnitude to and opposite direction to the force, F_{21}, exerted on object 1 by object 2.

Applications of Newton's Laws

Suppose you are pulling a box on frictionless ice, using a rope.



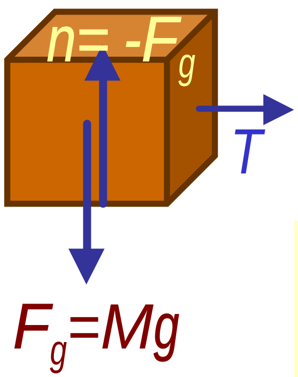
What are the forces being exerted on the box?

Gravitational force: F_g

Normal force: n

Tension force: T

Free-body diagram



Total force: $F = F_g + n + T = T$

$$\sum F_x = T = Ma_x; \quad a_x = \frac{T}{M}$$

$$\sum F_y = F_g + n = 0; \quad a_y = 0$$

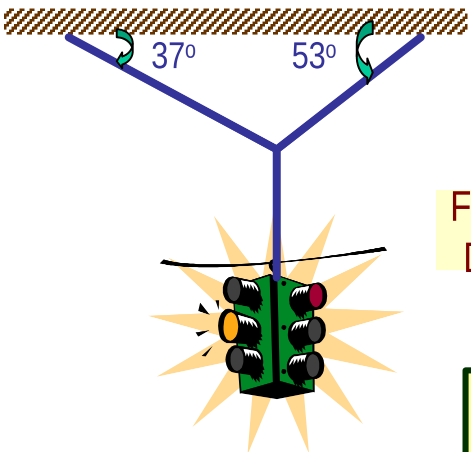
If T is a constant force, a_x, is constant

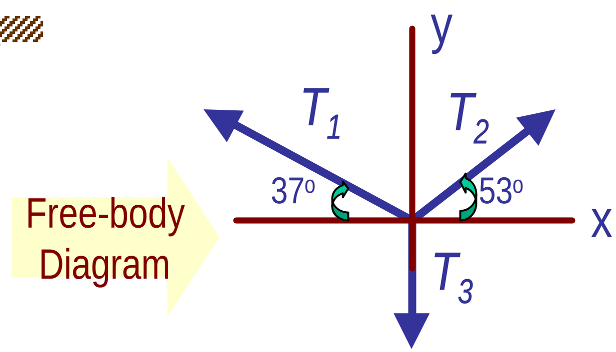
$$v_{xf} = v_{xi} + a_x t = v_{xi} + \left(\frac{T}{M}\right)t$$

$$\Delta x = x_f - x_i = v_{xi} t + \frac{1}{2}\left(\frac{T}{M}\right)t^2$$

Example 5.4

A traffic light weighing 125 N hangs from a cable tied to two other cables fastened to a support. The upper cables make angles of 37.0º and 53.0º with the horizontal. Find the tension in the three cables.





$$\vec{F} = \vec{T_1} + \vec{T_2} + \vec{T_3}; \quad F_x = \sum_{i=1}^{i=3} T_{ix} = 0; \quad F_y = \sum_{i=1}^{i=3} T_{iy} = 0$$

$$T_1 \sin(37^\circ) + T_2 \sin(53^\circ) - mg = 0$$

$$-T_1 \cos(37^\circ) + T_2 \cos(53^\circ) = 0$$

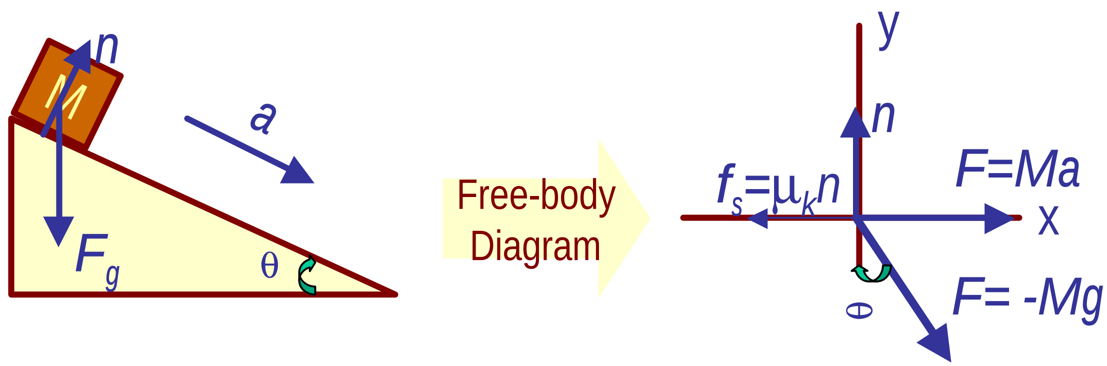
$$\therefore T_1 = \frac{\cos(53^\circ)}{\cos(37^\circ)} T_2 = 0.754\, T_2$$

$$T_2 \left[\sin(53^\circ) + 0.754 \times \sin(37^\circ)\right] = 1.25\, T_2 = 125\ N$$

$$T_2 = 100\ N; \quad T_1 = 0.754\, T_2 = 75.4\ N$$

Example 5.12

Suppose a block is placed on a rough surface inclined relative to the horizontal. The inclination angle is increased till the block starts to move. Show that by measuring this critical angle, θ_c, one can determine coefficient of static friction, μ_s.



$$\vec{F} = \vec{F_g} + \vec{n} + f_s$$

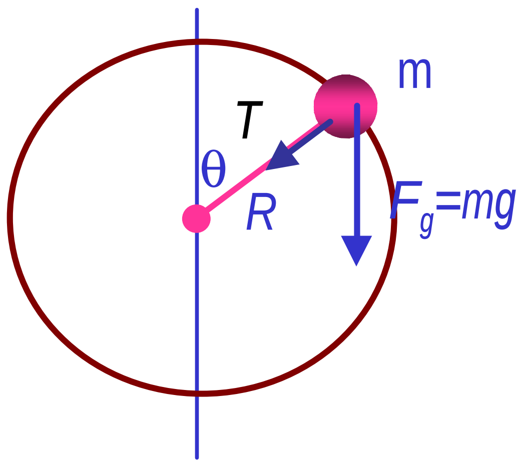
$$F_y = Ma_y = n + F_{gy} = 0; \quad n = -F_{gy} = Mg\cos q_c$$

$$F_x = F_{gx} - f_s = 0; \quad f_s = m_s n = Mg\sin q_c$$

$$m_s = \frac{Mg\sin q_c}{n} = \frac{Mg\sin q_c}{Mg\cos q_c} = \tan q_c$$

Example 6.8

A ball of mass m is attached to the end of a cord of length R. The ball is moving in a vertical circle. Determine the tension of the cord at any instant when the speed of the ball is *v* and the cord makes an angle θ with vertical.



What are the forces involved in this motion?

The gravitational force F_g and the radial force, T, providing tension.

$$\sum F_t = ma_t = mg\sin\theta$$

$$a_t = g\sin\theta$$

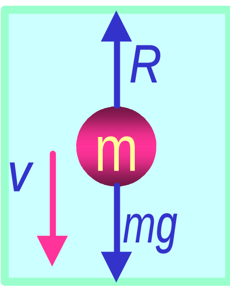
$$\sum F_r = T - mg\cos\theta = ma_r = m\frac{v^2}{R}$$

$$T = m\left(\frac{v^2}{R} + g\cos\theta\right)$$

At what angles the tension becomes maximum and minimum. What are the tension?

Example 6.11

A small ball of mass 2.00g is released from rest in a large vessel filled with oil, where it experiences a resistive force proportional to its speed. The ball reaches a terminal speed of 5.00 cm/s. Determine the time constant τ and the time it takes the ball to reach 90% of its terminal speed.



Determine the time constant τ.

$$v_t = \frac{mg}{b}$$

$$\therefore b = \frac{mg}{v_t} = \frac{2.00\times10^{-3}kg\cdot 9.80m/s^2}{5.00\times10^{-2}m/s} = 0.392kg/s$$

$$\tau = \frac{m}{b} = \frac{2.00\times10^{-3}kg}{0.392kg/s} = 5.10\times10^{-3}s$$

Determine the time it takes the ball to reach 90% of its terminal speed.

$$v = \frac{mg}{b}\left(1-e^{-t/\tau}\right) = v_t\left(1-e^{-t/\tau}\right)$$

$$0.9v_t = v_t\left(1-e^{-t/\tau}\right)$$

$$\left(1-e^{-t/\tau}\right) = 0.9;\ e^{-t/\tau} = 0.1$$

$$t = -\tau\cdot\ln 0.1 = 2.30\tau = 2.30\cdot 5.10\times10^{-3} = 11.7(ms)$$

Work and Kinetic Energy

Work in physics is done only when a sum of forces exerted on an object made a motion to the object.

What does this mean?

However much tired your arms feel, if you were just holding an object without moving it you have not done any physical work.

Mathematically, work is written in scalar product of force vector and the displacement vector

$$W = \sum \vec{F}_i \cdot \vec{d} = Fd\cos\theta$$

Kinetic Energy is the energy associated with motion and capacity to perform work. Work requires change of energy after the completion ← Work-Kinetic energy theorem

$$K = \frac{1}{2}mv^2$$

N.m=Joule

$$\sum W = K_f - K_i = \Delta K$$

Power is the rate of which work is performed.

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d}{dt}\left(\vec{s}\right) = \vec{F} \cdot \vec{v}$$

Units of these quantities????

Nm/s=Joule/s=Watt

Example 7.14

A compact car has a mass of 800kg, and its efficiency is rated at 18%. Find the amount of gasoline used to accelerate the car from rest to 27m/s (~60mi/h). Use the fact that the energy equivalent of 1gal of gasoline is 1.3×10^8J.

First let's compute what the kinetic energy needed to accelerate the car from rest to a speed *v*.

$$K_f = \frac{1}{2}mv^2 = \frac{1}{2} \times 800 \times (27)^2 = 2.9 \times 10^5\, J$$

Since the engine is only 18% efficient we must divide the necessary kinetic energy with this efficiency in order to figure out what the total energy needed is.

$$W_E = \frac{K_f}{e} = \frac{1}{2e}mv^2 = \frac{2.9 \times 10^5\, J}{0.18} = 16 \times 10^5\, J$$

Then using the fact that 1gal of gasoline can putout 1.3×10^8J, we can compute the total volume of gasoline needed to accelerate the car to 60 mi/h.

$$V_{gas} = \frac{W_E}{1.3 \times 10^8\, J/gal} = \frac{16 \times 10^5\, J}{1.3 \times 10^8\, J/gal} = 0.012\, gal$$

Potential Energy

Energy associated with a system of objects → Stored energy which has Potential or possibility to work or to convert to kinetic energy

What does this mean?

In order to describe potential energy, U, a system must be defined.

The concept of potential energy can only be used under the special class of forces called, conservative forces which results in principle of <u>conservation of mechanical energy.</u>

What other forms of energies in the universe?

Mechanical Energy

Chemical Energy

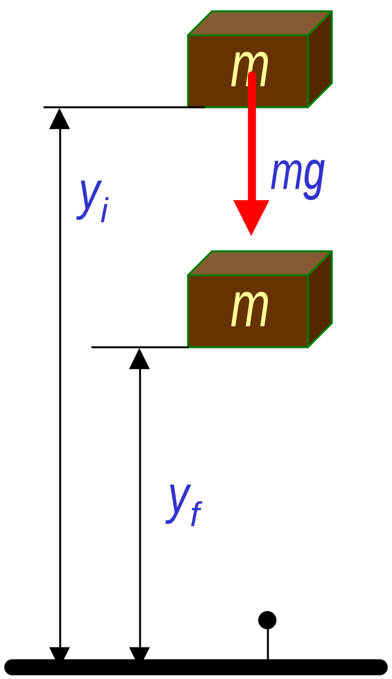
Biological Energy

Electromagnetic Energy

Nuclear Energy

Gravitational Potential

Potential energy given to an object by gravitational field in the system of Earth due to its height from the surface



When an object is falling, gravitational force, Mg, performs work on the object, increasing its kinetic energy. The potential energy of an object at a height y which is the potential to work is expressed as

$$U_g = \vec{F}_g \cdot \vec{y} = mg\left(-\vec{j}\right) \cdot y\left(-\vec{j}\right) = mgy$$

$$U_g \equiv mgy$$

Work performed on the object by the gravitational force as the brick goes from y_i to y_f is:

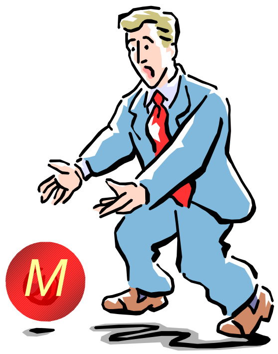
$$W_g = U_i - U_f = mgy_i - mgy_f = -\Delta U_g$$

What does this mean?

Work by the gravitational force as the brick goes from y_i to y_f is negative of the change in the system's potential energy

Example 8.1

A bowler drops bowling ball of mass 7kg on his toe. Choosing floor level as y=0, estimate the total work done on the ball by the gravitational force as the ball falls.



Let's assume the top of the toe is 0.03m from the floor and the hand was 0.5m above the floor.

$$U_i = mgy_i = 7 \times 9.8 \times 0.5 = 34.3\,J$$
$$U_f = mgy_f = 7 \times 9.8 \times 0.03 = 2.06\,J$$
$$\Delta U = -(U_f - U_i) = 32.24\,J \cong 30\,J$$

b) Perform the same calculation using the top of the bowler's head as the origin.

What has to change?

First we must re-compute the positions of ball at the hand and of the toe.

Assuming the bowler's height is 1.8m, the ball's original position is –1.3m, and the toe is at –1.77m.

$$U_i = mgy_i = 7 \times 9.8 \times (-1.3) = -89.2\,J$$
$$U_f = mgy_f = 7 \times 9.8 \times (-1.77) = -121.4\,J$$
$$\Delta U = -(U_f - U_i) = 32.2\,J \cong 30\,J$$

Elastic Potential Energy

Potential energy given to an object by a spring or an object with elasticity in the system consists of the object and the spring without friction.

The force spring exerts on an object when it is distorted from its equilibrium by a distance x is

$$F_s = -kx$$

The work performed on the object by the spring is

$$W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$$

The potential energy of this system is

$$U_s \equiv \frac{1}{2}kx^2$$

What do you see from the above equations?

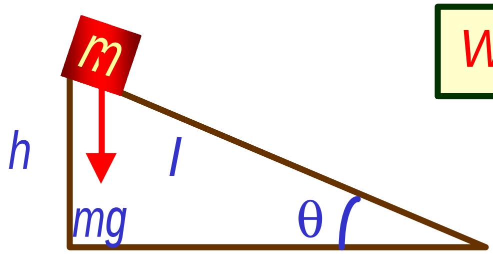
The work done on the object by the spring depends only on the initial and final position of the distorted spring.

Where else did you see this trend?

The gravitational potential energy, U_g

Conservative and Non-conservative Forces

The work done on an object by the gravitational force does not depend on the object's path.



When directly falls, the work done on the object is

$$W_g = mgh$$

When sliding down the hill of length l, the work is

$$W_g = F_{g-incline} \times l = mg \sin\theta \times l$$
$$= mg\left(l \sin\theta\right) = mgh$$

How about if we lengthen the incline by a factor of 2, keeping the height the same??

Still the same amount of work☺

$$W_g = mgh$$

So the work done by the gravitational force on an object is independent on the path of the object's movements. It only depends on the difference of the object's initial and final position in the direction of the force.

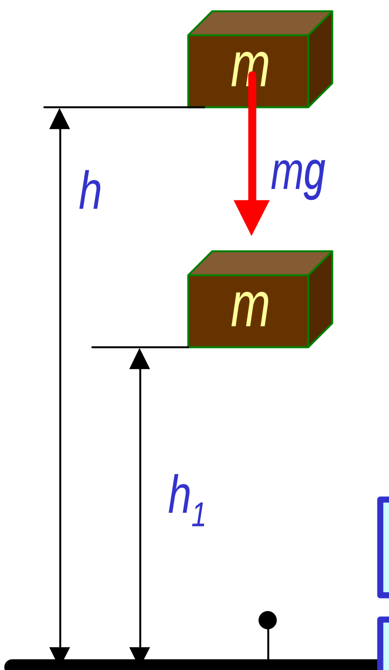
The forces like gravitational or elastic forces are called conservative forces

1. *If the work performed by the force does not depend on the path*
2. *If the work performed on a closed path is 0.*

Conservation of Mechanical Energy

Total mechanical energy is the sum of kinetic and potential energies

$$E \equiv K + U$$



Let's consider a brick of mass *m* at a height *h* from the ground

What is its potential energy?

$$U_g = mgh$$

What happens to the energy as the brick falls to the ground?

$$\Delta U = U_f - U_i = -\int_{x_i}^{x_f} F_x dx$$

The brick gains speed

By how much?

$$v = gt$$

So what?

The brick's kinetic energy increased

$$K = \frac{1}{2}mv^2 = \frac{1}{2}mg^2t^2$$

And?

The lost potential energy is converted to kinetic energy

What does this mean?

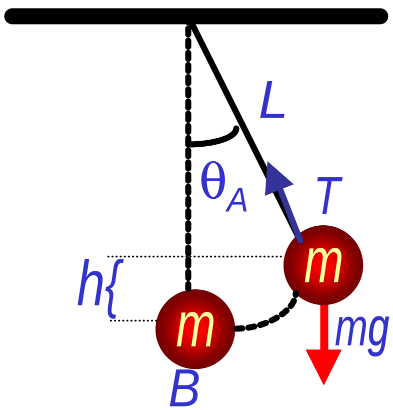
The total mechanical energy of a system remains constant in any isolated system of objects that interacts only through conservative forces: ***Principle of mechanical energy conservation***

$$E_i = E_f$$

$$K_i + \sum U_i = K_f + \sum U_f$$

Example 8.3

A ball of mass *m* is attached to a light cord of length L, making up a pendulum. The ball is released from rest when the cord makes an angle θ_A with the vertical, and the pivoting point P is frictionless. Find the speed of the ball when it is at the lowest point, B.



Compute the potential energy at the maximum height, *h*. Remember where the 0 is.

$$h = L - L\cos\theta_A$$
$$U_i = mgL(1 - \cos\theta_A)$$

Using the principle of mechanical energy conservation

$$K_i + U_i = K_f + U_f$$
$$mgh = mgL(1 - \cos\theta_A) = \frac{1}{2}mv^2$$
$$v^2 = 2gL(1 - \cos\theta_A)$$
$$\therefore v = \sqrt{2gL(1 - \cos\theta_A)}$$

b) Determine tension T at the point B.

Recall the centripetal acceleration of a circular motion

$$F_r = ma_r = T - mg = m\frac{v^2}{L}$$
$$T = m\left(g + \frac{v^2}{L}\right) = m\left(g + \frac{2gL(1 - \cos\theta_A)}{L}\right)$$
$$\therefore T = mg(3 - 2\cos\theta_A)$$

Cross check the result in a simple situation. What happens when the initial angle θ_A is 0? $T = mg$

Linear Momentum

The principle of energy conservation can be used to solve problems that are harder to solve just using Newton's laws. It is used to describe motion of an object or a system of objects.

A new concept of linear momentum can also be used to solve physical problems, especially the problems involving collisions of objects.

Linear momentum of an object whose mass is m and is moving at a velocity v is defined as

$$\boxed{\vec{p} \equiv m\vec{v}}$$

What can you tell from this definition about momentum?

1. *Momentum is a vector quantity.*
2. *The heavier the object the higher the momentum*
3. *The higher the velocity the higher the momentum*
4. *Its unit is kg.m/s*

What else can use see from the definition? Do you see force?

The change of momentum in a given time interval

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\left(m\vec{v}\right) = m\frac{d\vec{v}}{dt} = m\vec{a} = \vec{F}$$

Linear Momentum and Forces

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(m\vec{v}\right)$$

What can we learn from this Force-momentum relationship?

- *The rate of the change of particle's momentum is the same as the net force exerted on it.*
- *When net force is 0, the particle's linear momentum is constant.*
- *If a particle is isolated, the particle experiences no net force, therefore its momentum does not change and is conserved.*

Something else we can do with this relationship. What do you think it is?

The relationship can be used to study the case where the mass changes as a function of time.

Can you think of a few cases like this?

Motion of a meteorite

Trajectory a satellite

Conservation of Linear Momentum in a Two Particle System

Consider a system with two particles that does not have any external forces exerted on it. What is the impact of Newton's 3rd Law?

If particle#1 exerts force on particle #2, there must be another force that the particle #2 exerts on #1 as the reaction force. Both the forces are internal forces and the net force in the SYSTEM is still 0.

Now how would the momenta of these particles look like?

Let say that the particle #1 has momentum p_1 and #2 has p_2 at some point of time.

Using momentum-force relationship

$$\vec{F}_{21} = \frac{d\vec{p}_1}{dt} \quad \text{and} \quad \vec{F}_{12} = \frac{d\vec{p}_2}{dt}$$

And since net force of this system is 0

$$\sum \vec{F} = \vec{F}_{12} + \vec{F}_{21} = \frac{d\vec{p}_2}{dt} + \frac{d\vec{p}_1}{dt} = \frac{d}{dt}\left(\vec{p}_2 + \vec{p}_1\right) = 0$$

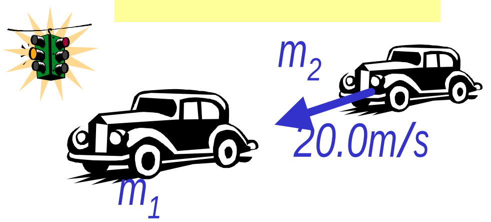
Therefore $\vec{p}_2 + \vec{p}_1 = const$

The total linear momentum of the system is conserved!!!

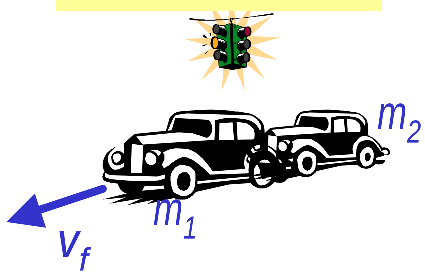
Example 9.5

A car of mass 1800kg stopped at a traffic light is rear-ended by a 900kg car, and the two become entangled. If the lighter car was moving at 20.0m/s before the collision what is the velocity of the entangled cars after the collision?

Before collision



After collision



The momenta before and after the collision are

$$\vec{p}_i = m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_2\vec{v}_{2i}$$

$$\vec{p}_f = m_1\vec{v}_{1f} + m_2\vec{v}_{2f} = (m_1 + m_2)\vec{v}_f$$

Since momentum of the system must be conserved

$$\vec{p}_i = \vec{p}_f$$

$$(m_1 + m_2)\vec{v}_f = m_2\vec{v}_{2i}$$

$$\vec{v}_f = \frac{m_2\vec{v}_{2i}}{(m_1 + m_2)} = \frac{900 \times 20.0\vec{i}}{900 + 1800} = 6.67\vec{i}\ m/s$$

What can we learn from these equations on the direction and magnitude of the velocity before and after the collision?

The cars are moving in the same direction as the lighter car's original direction to conserve momentum.

The magnitude is inversely proportional to its own mass.

Elastic and Inelastic Collisions

Momentum is conserved in any collisions as long as external forces negligible.

Collisions are classified as elastic or inelastic by the conservation of kinetic energy before and after the collisions.

Elastic Collision	A collision in which the total kinetic energy is the same before and after the collision.
Inelastic Collision	A collision in which the total kinetic energy is not the same before and after the collision.

Two types of inelastic collisions:Perfectly inelastic and inelastic

Perfectly Inelastic: Two objects stick together after the collision moving at a certain velocity together.

Inelastic: Colliding objects do not stick together after the collision but some kinetic energy is lost.

Note: Momentum is constant in all collisions but kinetic energy is only in elastic collisions.

Elastic and Perfectly Inelastic Collisions

In perfectly Inelastic collisions, the objects stick together after the collision, moving together. Momentum is conserved in this collision, so the final velocity of the stuck system is

$$m_1\overrightarrow{v_{1i}} + m_2\overrightarrow{v_{2i}} = (m_1 + m_2)\overrightarrow{v_f}$$

$$\overrightarrow{v_f} = \frac{m_1\overrightarrow{v_{1i}} + m_2\overrightarrow{v_{2i}}}{(m_1 + m_2)}$$

How about elastic collisions?

In elastic collisions, both the momentum and the kinetic energy are conserved. Therefore, the final speeds in an elastic collision can be obtained in terms of initial speeds as

$$m_1\overrightarrow{v_{1i}} + m_2\overrightarrow{v_{2i}} = m_1\overrightarrow{v_{1f}} + m_2\overrightarrow{v_{2f}}$$

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$

$$m_1\left(v_{1i}^2 - v_{1f}^2\right) = m_2\left(v_{2i}^2 - v_{2f}^2\right)$$

$$m_1\left(v_{1i} - v_{1f}\right)\left(v_{1i} + v_{1f}\right) = m_2\left(v_{2i} - v_{2f}\right)\left(v_{2i} + v_{2f}\right)$$

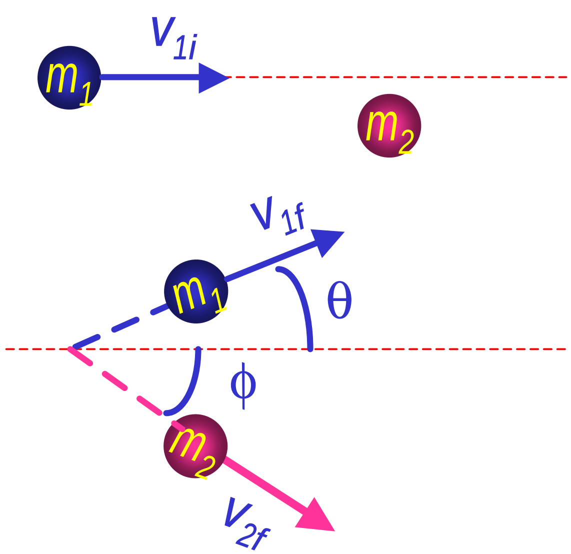
From 1-dim momentum conservation:

$$m_1\left(v_{1i} - v_{1f}\right) = m_2\left(v_{2i} - v_{2f}\right)$$

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i}; \quad v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} + \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{2i}$$

Example 9.9

Proton #1 with a speed 3.50×10^5 m/s collides elastically with proton #2 initially at rest. After the collision, proton #1 moves at an angle of 37° to the horizontal axis and proton #2 deflects at an angle ϕ to the same axis. Find the final speeds of the two protons and the scattering angle of proton #2, ϕ.



Since both the particles are protons $m_1=m_2=m_p$.

Using momentum conservation, one obtains

$$m_p v_{1i} = m_p v_{1f}\cos\theta + m_p v_{2f}\cos\phi$$
$$m_p v_{1f}\sin\theta - m_p v_{2f}\sin\phi = 0$$

Canceling m_p and put in all known quantities, one obtains

$$v_{1f}\cos 37^\circ + v_{2f}\cos\phi = 3.50\times10^5 \quad (1)$$
$$v_{1f}\sin 37^\circ = v_{2f}\sin\phi \quad (2)$$

From kinetic energy conservation:

$$\left(3.50\times10^5\right)^2 = v_{1f}^2 + v_{2f}^2 \quad (3)$$

Solving Eqs. 1-3 equations, one gets

$$v_{1f} = 2.80\times10^5\ m/s$$
$$v_{2f} = 2.11\times10^5\ m/s$$
$$\phi = 53.0^\circ$$

Center of Mass of a Rigid Object

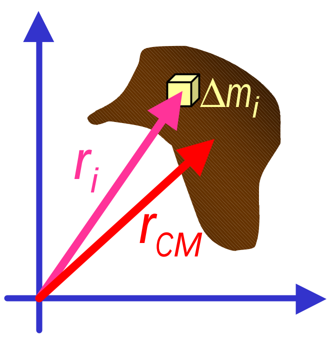
The formula for CM can be expanded to Rigid Object or a system of many particles

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_i m_ix_i}{\sum_i m_i}$$

$$y_{CM} = \frac{\sum_i m_iy_i}{\sum_i m_i}; \qquad z_{CM} = \frac{\sum_i m_iz_i}{\sum_i m_i}$$

The position vector of the center of mass of a many particle system is

$$\vec{r}_{CM} = x_{CM}\vec{i} + y_{CM}\vec{j} + z_{CM}\vec{k}$$
$$= \frac{\sum_i m_ix_i\vec{i} + \sum_i m_iy_i\vec{j} + \sum_i m_iz_i\vec{k}}{\sum_i m_i} = \frac{\sum_i m_i\vec{r}_i}{M}$$



A rigid body – an object with shape and size with mass spread throughout the body, ordinary objects – can be considered as a group of particles with mass m_i densely spread throughout the given shape of the object

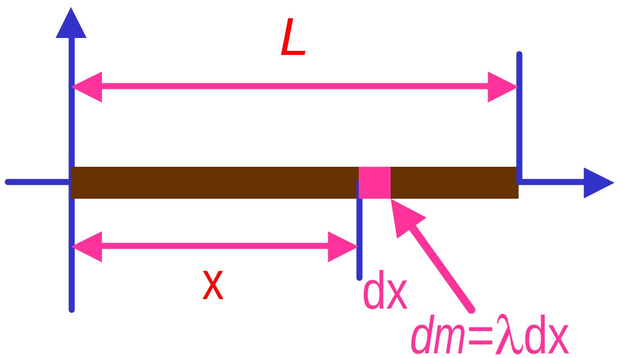
$$x_{CM} \approx \frac{\sum_i \Delta m_ix_i}{M}$$

$$x_{CM} = \lim_{\Delta m_i \to 0} \frac{\sum_i \Delta m_ix_i}{M} = \frac{1}{M}\int xdm$$

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}\,dm$$

Example 9.13

Show that the center of mass of a rod of mass *M* and length *L* lies in midway between its ends, assuming the rod has a uniform mass per unit length.



The formula for CM of a continuous object is

$$x_{CM} = \frac{1}{M}\int_{x=0}^{x=L} x\,dm$$

Since the density of the rod is constant, one can write

$$dm = l\,dx;\ \text{where}\ \ l = M/L$$

Therefore

$$x_{CM} = \frac{1}{M}\int_{x=0}^{x=L} l\,x\,dx = \frac{1}{M}\left[\frac{1}{2}l\,x^2\right]_{x=0}^{x=L} = \frac{1}{M}\left(\frac{1}{2}l\,L^2\right) = \frac{1}{M}\left(\frac{1}{2}ML\right) = \frac{L}{2}$$

Find the CM when the density of the rod non-uniform but varies linearly as a function of x, λ=α x

$$M = \int_{x=0}^{x=L} l\,dx = \int_{x=0}^{x=L} a\,x\,dx$$
$$= \left[\frac{1}{2}a\,x^2\right]_{x=0}^{x=L} = \frac{1}{2}a\,L^2$$

$$x_{CM} = \frac{1}{M}\int_{x=0}^{x=L} l\,x\,dx = \frac{1}{M}\int_{x=0}^{x=L} a\,x^2\,dx = \frac{1}{M}\left[\frac{1}{3}a\,x^3\right]_{x=0}^{x=L}$$
$$= \frac{1}{M}\left(\frac{1}{3}a\,L^3\right) = \frac{1}{M}\left(\frac{2}{3}ML\right) = \frac{2L}{3}$$

Motion of a Group of Particles

We've learned that the CM of a system can represent the motion of a system. Therefore, for an isolated system of many particles in which the total mass *M* is preserved, the velocity, total momentum, acceleration of the system are

Velocity of the system

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{d}{dt}\left(\frac{1}{M}\sum m_i \vec{r}_i\right) = \frac{1}{M}\sum m_i \frac{d\vec{r}_i}{dt} = \frac{\sum m_i \vec{v}_i}{M}$$

Total Momentum of the system

$$\vec{p}_{tot} = M\vec{v}_{CM} = M\frac{\sum m_i \vec{v}_i}{M} = \sum m_i \vec{v}_i = \sum \vec{p}_i$$

Acceleration of the system

$$\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{d}{dt}\left(\frac{1}{M}\sum m_i \vec{v}_i\right) = \frac{1}{M}\sum m_i \frac{d\vec{v}_i}{dt} = \frac{\sum m_i \vec{a}_i}{M}$$

External force exerting on the system

$$\sum \vec{F}_{ext} = M\vec{a}_{CM} = \sum m_i \vec{a}_i = \frac{d\vec{p}_{tot}}{dt}$$

What about the internal forces?

If net external force is 0

$$\sum \vec{F}_{ext} = 0 = \frac{d\vec{p}_{tot}}{dt}; \quad \vec{p}_{tot} = \text{const}$$

System's momentum is conserved.

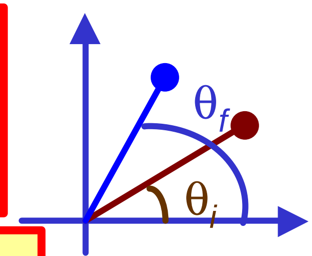
Angular Displacement, Velocity, and Acceleration

Using what we have learned in the previous slide, how would you define the angular displacement?

$$\Delta q = q_f - q_i$$

How about the average angular speed?

$$\overline{w} \equiv \frac{q_f - q_i}{t_f - t_i} = \frac{\Delta q}{\Delta t}$$



And the instantaneous angular speed?

$$w \equiv \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t} = \frac{dq}{dt}$$

By the same token, the average angular acceleration

$$\overline{a} \equiv \frac{w_f - w_i}{t_f - t_i} = \frac{\Delta w}{\Delta t}$$

And the instantaneous angular acceleration?

$$a \equiv \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = \frac{dw}{dt}$$

When rotating about a fixed axis, every particle on a rigid object rotates through the same angle and has the same angular speed and angular acceleration.

Rotational Kinematics

The first type of motion we have learned in linear kinematics was under a constant acceleration. We will learn about the rotational motion under constant acceleration, because these are the simplest motions in both cases.

Just like the case in linear motion, one can obtain

Angular Speed under constant angular acceleration:

$$\boxed{w_f = w_i + at}$$

Angular displacement under constant angular acceleration:

$$\boxed{q_f = q_i + w_i t + \frac{1}{2} a t^2}$$

One can also obtain

$$\boxed{w_f^2 = w_i^2 + 2a\left(q_f - q_i\right)}$$

Example 10.1

A wheel rotates with a constant angular acceleration pf 3.50 rad/s^2. If the angular speed of the wheel is 2.00 rad/s at t_i=0, a) through what angle does the wheel rotate in 2.00s?

Using the angular displacement formula in the previous slide, one gets

$$q_f - q_i = wt + \frac{1}{2}at^2$$
$$= 2.00 \times 2.00 + \frac{1}{2}3.50 \times (2.00)^2$$
$$= 11.0\,rad \quad = \frac{11.0}{2p} rev. = 1.75\,rev.$$

What is the angular speed at t=2.00s?

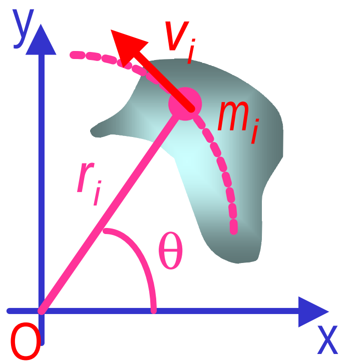
Using the angular speed and acceleration relationship

$$w_f = w_i + at$$
$$= 2.00 + 3.50 \times 2.00$$
$$= 9.00\,rad/s$$

Find the angle through which the wheel rotates between t=2.00 s and t=3.00 s.

$$q_2 = 11.0\,rad$$
$$q_3 = 2.00 \times 3.00 + \frac{1}{2}3.50 \times (3.00)^2 = 21.8\,rad$$
$$\Delta q = q_3 - q_2 = 10.8\,rad \quad = \frac{10.8}{2p} rev. = 1.72\,rev.$$

Rotational Energy



What do you think the kinetic energy of a rigid object that is undergoing a circular motion is?

Kinetic energy of a masslet, m_i, moving at a tangential speed, v_i, is

$$K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 w^2$$

Since a rigid body is a collection of masslets, the total kinetic energy of the rigid object is

$$K_R = \sum_i K_i = \frac{1}{2} \sum_i m_i r_i^2 w^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right) w^2$$

By defining a new quantity called, Moment of Inertia, I, as

$$I = \sum_i m_i r_i^2$$

The above expression is simplified as

$$K_R = \frac{1}{2} I w^2$$

What are the dimension and unit of Moment of Inertia?

$$kg \cdot m^2 \qquad \left[ML^2\right]$$

What do you think the moment of inertia is?

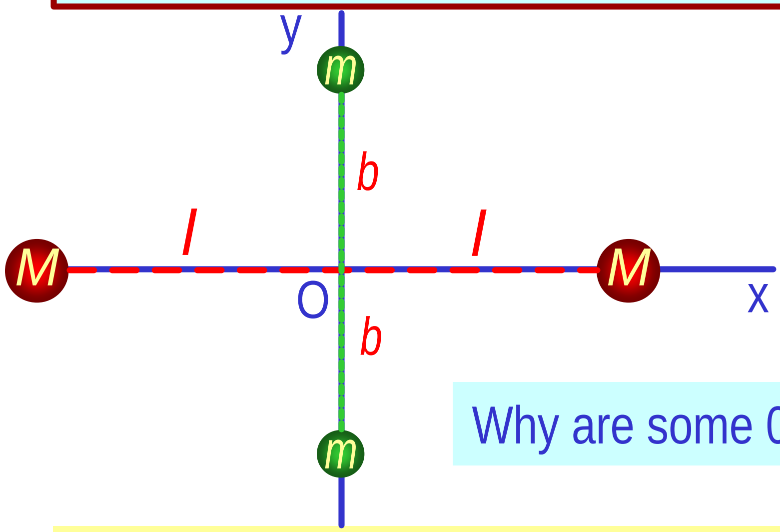
Measure of resistance of an object to changes in its rotational motion.

What similarity do you see between rotational and linear kinetic energies?

Mass and speed in linear kinetic energy are replaced by moment of inertia and angular speed.

Example 10.4

In a system consists of four small spheres as shown in the figure, assuming the radii are negligible and the rods connecting the particles are massless, compute the moment of inertia and the rotational kinetic energy when the system rotates about the y-axis at ω.



Since the rotation is about y axis, the moment of inertia about y axis, I_y, is

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + m \cdot 0^2 + m \cdot 0^2 = 2Ml^2$$

Why are some 0s?

This is because the rotation is done about y axis, and the radii of the spheres are negligible.

Thus, the rotational kinetic energy is

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2\right)\omega^2 = Ml^2\omega^2$$

Find the moment of inertia and rotational kinetic energy when the system rotates on the x-y plane about the z-axis that goes through the origin O.

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + mb^2 + mb^2 = 2\left(Ml^2 + mb^2\right)$$

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2 + 2mb^2\right)\omega^2 = \left(Ml^2 + mb^2\right)\omega^2$$

Calculation of Moments of Inertia

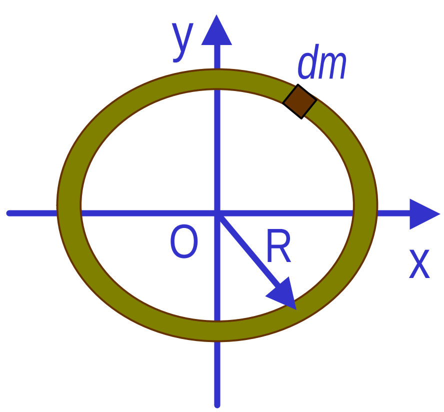
Moments of inertia for large objects can be computed, if we assume the object consists of small volume elements with mass, Δm_i.

The moment of inertia for the large rigid object $I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$

It is sometimes easier to compute moments of inertia in terms of volume of the elements rather than their mass

Using the volume density, ρ, replace d*m* in the above equation with dV. $\rho = \frac{dm}{dV}; \quad dm = \rho dV$ The moments of inertia becomes $I = \int \rho r^2 dV$

Example 10.5: Find the moment of inertia of a uniform hoop of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center.



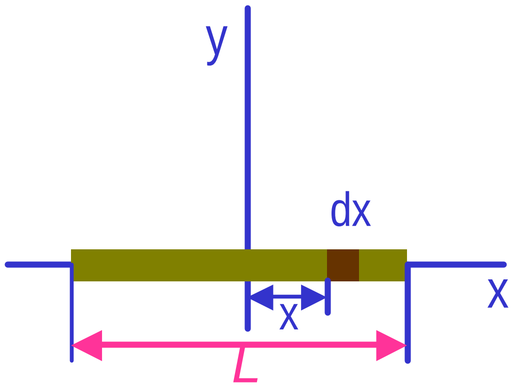
The moment of inertia is $I = \int r^2 dm = R^2 \int dm = MR^2$

What do you notice from this result?

The moment of inertia for this object is the same as that of a point of mass M at the distance R.

Example 10.6

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod and passing through its center of mass.



The line density of the rod is

$$l = \frac{M}{L}$$

so the masslet is

$$dm = l\,dx = \frac{M}{L}dx$$

The moment of inertia is

$$I = \int r^2 dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

What is the moment of inertia when the rotational axis is at one end of the rod.

$$I = \int r^2 dm = \int_0^L \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_0^L$$

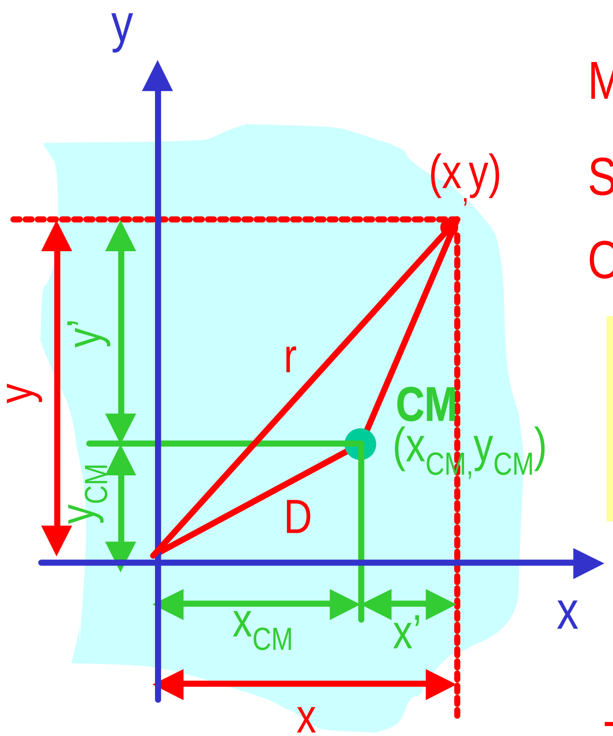
$$= \frac{M}{3L}\left[(L)^3 - 0\right] = \frac{M}{3L}\left(L^3\right) = \frac{ML^2}{3}$$

Will this be the same as the above. Why or why not?

Since the moment of inertia is resistance to motion, it makes perfect sense for it to be harder to move when it is rotating about the axis at one end.

Parallel Axis Theorem

Moments of inertia for highly symmetric object is relatively easy if the rotational axis is the same as the axis of symmetry. However if the axis of rotation does not coincide with axis of symmetry, the calculation can still be done in simple manner using **parallel-axis theorem**. $I = I_{CM} + MD^2$



Moment of inertia is defined $I = \int r^2 dm = \int \sqrt{(x^2 + y^2)}dm \quad (1)$

Since x and y are $x = x_{CM} + x'; \quad y = y_{CM} + y'$

One can substitute x and y in Eq. 1 to obtain

$$I = \int \left[(x_{CM} + x')^2 + (y_{CM} + y')^2 \right] dm$$
$$= (x_{CM}^2 + y_{CM}^2) \int dm + 2x_{CM} \int x' dm + 2y_{CM} \int y' dm + \int (x'^2 + y'^2) dm$$

Since the x' and y' are the distance from CM, by definition

$$\int x' dm = 0$$
$$\int y' dm = 0$$

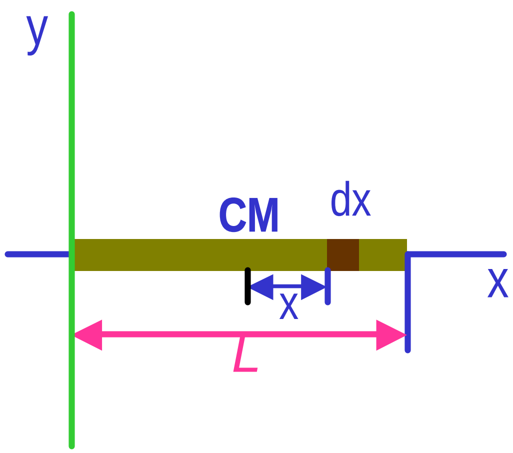
Therefore, the parallel-axis theorem $I = I_{CM} + MD^2$

What does this theorem tell you?

Moment of inertia of any object about any arbitrary axis are the same as the sum of moment of inertia for a rotation about the CM and that of the CM about the rotation axis.

Example 10.8

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis that goes through one end of the rod, using parallel-axis theorem.



The line density of the rod is $l = \frac{M}{L}$

so the masslet is $dm = l\,dx = \frac{M}{L}dx$

The moment of inertia about the CM

$$I_{CM} = \int r^2 dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

Using the parallel axis theorem

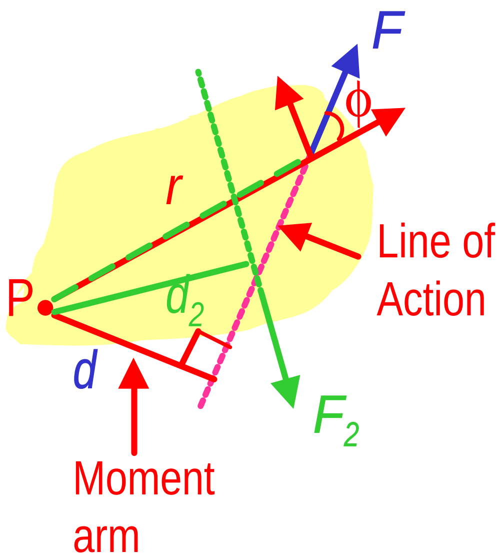
$$I = I_{CM} + D^2 M = \frac{ML^2}{12} + \left(\frac{L}{2}\right)^2 M = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}$$

The result is the same as using the definition of moment of inertia.

Parallel-axis theorem is useful to compute moment of inertia of a rotation of a rigid object with complicated shape about an arbitrary axis

Torque

Torque is the tendency of a force to rotate an object about some axis. Torque, t, is a vector quantity.



Consider an object pivoting about the point P by the force *F* being exerted at a distance r.

The line that extends out of the tail of the force vector is called the line of action.

The perpendicular distance from the pivoting point P to the line of action is called Moment arm.

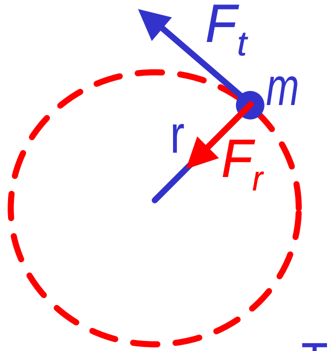
Magnitude of torque is defined as the product of the force exerted on the object to rotate it and the moment arm.

$$t \equiv rF\sin f = Fd$$

When there are more than one force being exerted on certain points of the object, one can sum up the torque generated by each force vectorially. The convention for sign of the torque is positive if rotation is in counter-clockwise and negative if clockwise.

$$\sum t = t_1 + t_2 = Fd - F_2 d_2$$

Torque & Angular Acceleration



Let's consider a point object with mass *m* rotating in a circle.

What forces do you see in this motion?

The tangential force F_t and radial force F_r

The tangential force F_t is $F_t = ma_t = mr\alpha$

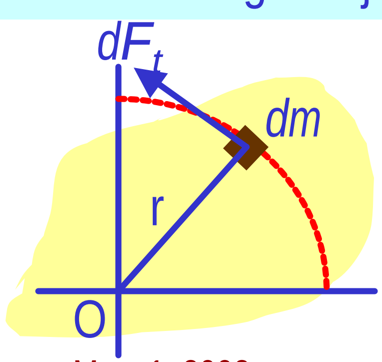
The torque due to tangential force F_t is $\tau = F_t r = ma_t r = mr^2\alpha$

What do you see from the above relationship? $\tau = I\alpha$

What does this mean? Torque acting on a particle is proportional to the angular acceleration.

What law do you see from this relationship? Analogs to Newton's 2nd law of motion in rotation.

How about a rigid object?



The external tangential force dF_t is $dF_t = dma_t = dmr\alpha$

The torque due to tangential force F_t is $d\tau = dF_t r = (r^2 dm)\alpha$

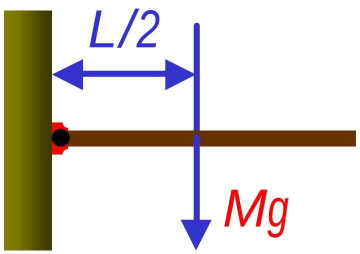
The total torque is $\sum \tau = \alpha \int r^2 dm = I\alpha$

What is the contribution due to radial force and why?

Contribution from radial force is 0, because its line of action passes through the pivoting point, making the moment arm 0.

Example 10.10

A uniform rod of length *L* and mass *M* is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane. The rod is released from rest in the horizontal position what is the initial angular acceleration of the rod and the initial linear acceleration of its right end?



The only force generating torque is the gravitational force *Mg*

$$t = Fd = F\frac{L}{2} = Mg\frac{L}{2} = Ia$$

Since the moment of inertia of the rod when it rotates about one end is

$$I = \int_0^L r^2 dm = \int_0^L x^2 l dx = \left(\frac{M}{L}\right)\left[\frac{x^3}{3}\right]_0^L = \frac{ML^2}{3}$$

We obtain

$$a = \frac{MgL}{2I} = \frac{MgL}{\frac{2ML^2}{3}} = \frac{3g}{2L}$$

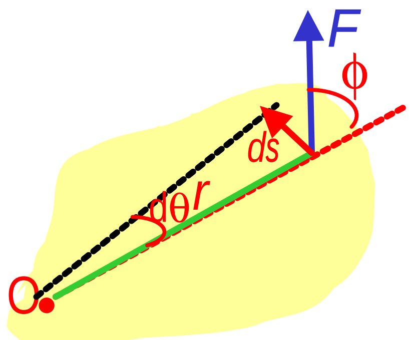
Using the relationship between tangential and angular acceleration

$$a_t = La = \frac{3g}{2}$$

What does this mean?

The tip of the rod falls faster than an object undergoing a free fall.

Work, Power, and Energy in Rotation



Let's consider a motion of a rigid body with a single external force *F* exerted on the point P, moving the object by *ds*.

The work done by the force *F* as the object rotates through infinitesimal distance ds=rdθ in a time dt is

$$dW = \vec{F}\cdot d\vec{s} = (F\sin\phi)rd\theta$$

What is *F*sinϕ? The tangential component of force *F*.

What is the work done by radial component *F*cosϕ? Zero, because it is perpendicular to the displacement.

Since the magnitude of torque is r*F*sinϕ,

$$dW = \tau d\theta$$

The rate of work, or power becomes

$$P = \frac{dW}{dt} = \frac{\tau d\theta}{dt} = \tau\omega$$

How was the power defined in linear motion?

The rotational work done by an external force equals the change in rotational energy.

$$\sum\tau = I\alpha = I\left(\frac{d\omega}{dt}\right) = I\left(\frac{d\omega}{d\theta}\right)\left(\frac{d\theta}{dt}\right)$$

The work put in by the external force then

$$dW = \sum\tau\, d\theta = I\omega d\omega$$

$$\sum W = \int_{\theta_i}^{\theta_i}\sum\tau\, d\theta = \int_{\omega_i}^{\omega_f} I\omega d\omega = \frac{1}{2}I\omega_i^2 - \frac{1}{2}I\omega_f^2$$

Similarity Between Linear and Rotational Motions

All physical quantities in linear and rotational motions show striking similarity.

Similar Quantity	Linear	Rotational
Mass	Mass M	Moment of Inertia $I = \int r^2 dm$
Length of motion	Distance L	Angle q (Radian)
Speed	$v = \frac{dr}{dt}$	$w = \frac{dq}{dt}$
Acceleration	$a = \frac{dv}{dt}$	$a = \frac{dw}{dt}$
Force	Force $F = ma$	Torque $t = Ia$
Work	Work $W = \int_{x_i}^{x_f} F dx$	Work $W = \int_{q_i}^{q_f} t\, dq$
Power	$P = \vec{F} \cdot \vec{v}$	$P = tw$
Momentum	$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{w}$
Kinetic Energy	Kinetic $K = \frac{1}{2} mv^2$	Rotational $K_R = \frac{1}{2} I w^2$

Rolling Motion of a Rigid Body

What is a rolling motion?

A more generalized case of a motion where the rotational axis moves together with the object

A rotational motion about the moving axis

To simplify the discussion, let's make a few assumptions

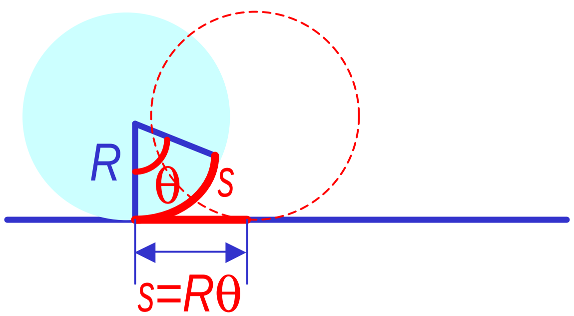
1. Limit our discussion on very symmetric objects, such as cylinders, spheres, etc
2. The object rolls on a flat surface

Let's consider a cylinder rolling without slipping on a flat surface

Under what condition does this "Pure Rolling" happen?

The total linear distance the CM of the cylinder moved is

$$s = R\theta$$



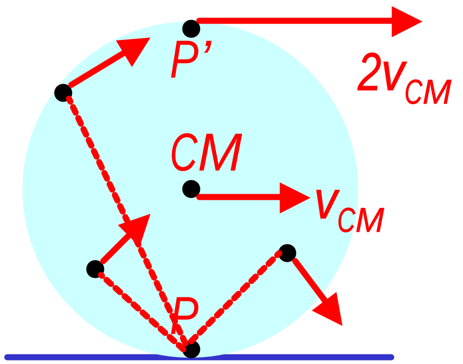
Thus the linear speed of the CM is

$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

Condition for "Pure Rolling"

Total Kinetic Energy of a Rolling Body

What do you think the total kinetic energy of the rolling cylinder is?



Since it is a rotational motion about the point P, we can writ the total kinetic energy

$$K = \frac{1}{2} I_P \omega^2$$

Where, I_P, is the moment of inertia about the point P.

Using the parallel axis theorem, we can rewrite

$$K = \frac{1}{2} I_P \omega^2 = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M R^2 \omega^2$$

Since $v_{CM}=R\omega$, the above relationship can be rewritten as

$$K = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M v_{CM}^2$$

Rotational kinetic energy about the CM

Translational Kinetic energy of the CM

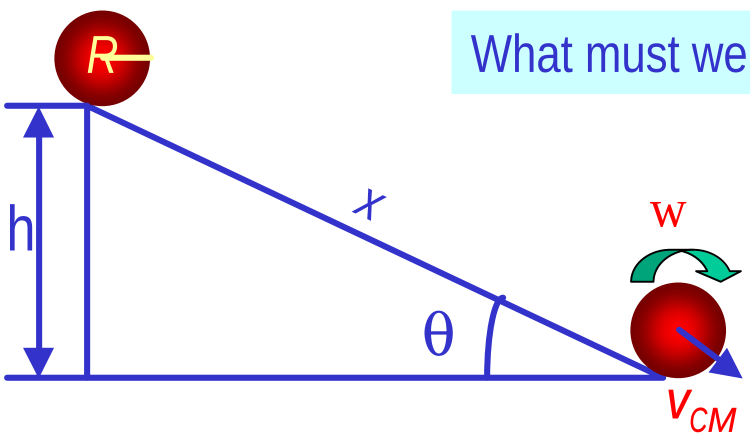
What does this equation mean?

Total kinetic energy of a rolling motion is the sum of the rotational kinetic energy about the CM

And the translational kinetic of the CM

Example 11.1

For solid sphere as shown in the figure, calculate the linear speed of the CM at the bottom of the hill and the magnitude of linear acceleration of the CM.



What must we know first?

The moment of inertia the sphere with respect to the CM!!

$$I_{CM} = \int r^2 dm = \frac{2}{5} MR^2$$

Thus using the formula in the previous slide

$$v_{CM} = \sqrt{\frac{2gh}{1 + I_{CM}/MR^2}} = \sqrt{\frac{2gh}{1+2/5}} = \sqrt{\frac{10}{7}gh}$$

Since h=xsinθ, one obtains

$$v_{CM}^2 = \frac{10}{7} gx \sin\theta$$

Using kinematic relationship

$$v_{CM}^2 = 2a_{CM}x$$

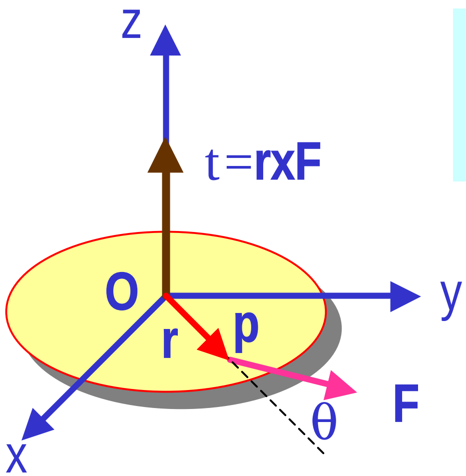
The linear acceleration of the CM is

$$a_{CM} = \frac{v_{CM}^2}{2x} = \frac{5}{7} g \sin\theta$$

What do you see?

Linear acceleration of a sphere does not depend on anything but g and θ.

Torque and Vector Product



Let's consider a disk fixed onto the origin O and the force *F* exerts on the point p. What happens?

The disk will start rotating counter clockwise about the Z axis

The magnitude of torque given to the disk by the force *F* is

$$\tau = Fr \sin \phi$$

But torque is a vector quantity, what is the direction?
How is torque expressed mathematically?

$$\boxed{\vec{\tau} \equiv \vec{r} \times \vec{F}}$$

What is the direction?

The direction of the torque follows the right-hand rule!!

The above quantity is called
Vector product or Cross product

$$\vec{C} \equiv \vec{A} \times \vec{B}$$

$$\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right| \sin \theta$$

What is the result of a vector product?

Another vector

What is another vector operation we've learned?

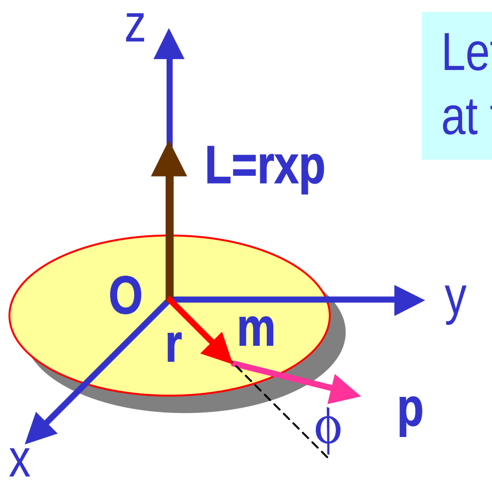
Scalar product

$$C \equiv \vec{A} \cdot \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right| \cos \theta$$

Result? A scalar

Angular Momentum of a Particle

If you grab onto a pole while running, your body will rotate about the pole, gaining angular momentum. We've used linear momentum to solve physical problems with linear motions, angular momentum will do the same for rotational motions.



Let's consider a point-like object (particle) with mass *m* located at the vector location ***r*** and moving with linear velocity ***v***

The instantaneous angular momentum ***L*** of this particle relative to origin O is

$$\boxed{\vec{L} \equiv \vec{r} \times \vec{p}}$$

What is the unit and dimension of angular momentum? $kg \cdot m^2 / s^2$

Note that ***L*** depends on origin O. Why? Because ***r*** changes

What else do you learn? The direction of ***L*** is +z

Since ***p*** is *m**v***, the magnitude of ***L*** becomes $L = mvr \sin f$

What do you learn from this?

If the direction of linear velocity points to the origin of rotation, the particle does not have any angular momentum.

If the linear velocity is perpendicular to position vector, the particle moves exactly the same way as a point on a rim.

The point O has to be inertial.

Angular Momentum and Torque

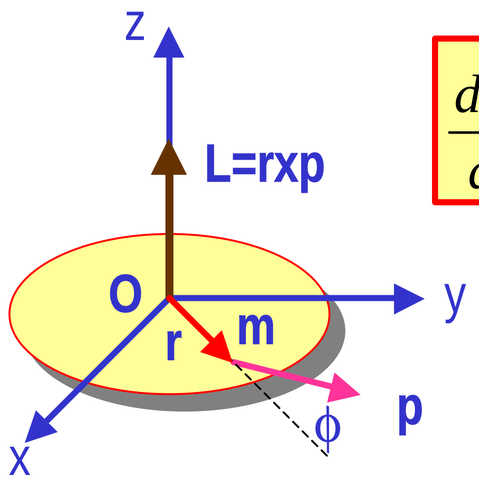
Can you remember how net force exerting on a particle and the change of its linear momentum are related?

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

Total external forces exerting on a particle is the same as the change of its linear momentum.

The same analogy works in rotational motion between torque and angular momentum.

Net torque acting on a particle is $\sum \vec{t} = \vec{r} \times \sum \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}$



$$\frac{d\vec{L}}{dt} = \frac{d\left(\vec{r} \times \vec{p}\right)}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = 0 + \vec{r} \times \frac{d\vec{p}}{dt}$$

Why does this work?

Because *v* is parallel to the linear momentum

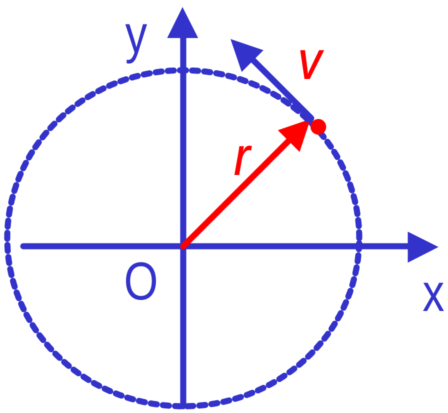
Thus the torque-angular momentum relationship

$$\sum \vec{t} = \frac{d\vec{L}}{dt}$$

The net torque acting on a particle is the same as the time rate change of its angular momentum

Example 11.4

A particle of mass *m* is moving in the xy plane in a circular path of radius r and linear velocity v about the origin O. Find the magnitude and direction of angular momentum with respect to O.



Using the definition of angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m\vec{r} \times \vec{v}$$

Since both the vectors, r and v, are on x-y plane and using right-hand rule, the direction of the angular momentum vector is +z (coming out of the screen)

The magnitude of the angular momentum is $\left|\vec{L}\right| = \left|m\vec{r} \times \vec{v}\right| = mrv \sin f = mrv \sin 90^\circ = mrv$

So the angular momentum vector can be expressed as $\vec{L} = mrv\vec{k}$

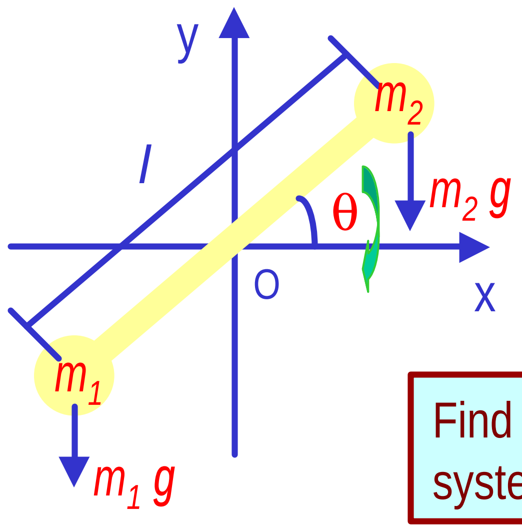
Find the angular momentum in terms of angular velocity w.

Using the relationship between linear and angular speed

$$\vec{L} = mrv\vec{k} = mr^2 w\vec{k} = mr^2\vec{w} = I\vec{w}$$

Example 11.6

A rigid rod of mass *M* and length *l* pivoted without friction at its center. Two particles of mass m_1 and m_2 are connected to its ends. The combination rotates in a vertical plane with an angular speed of ω. Find an expression for the magnitude of the angular momentum.



The moment of inertia of this system is

$$I = I_{rod} + I_{m_1} + I_{m_2} = \frac{1}{12}Ml^2 + \frac{1}{4}m_1 l^2 + \frac{1}{4}m_2 l^2$$

$$= \frac{l^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)$$

$$\boxed{L = Iw = \frac{wl^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)}$$

Find an expression for the magnitude of the angular acceleration of the system when the rod makes an angle θ with the horizon.

If $m_1 = m_2$, no angular momentum because net torque is 0.

If $\theta = +/-\pi/2$, at equilibrium so no angular momentum.

First compute net external torque

$$t_1 = m_1 g \frac{l}{2}\cos q\,;\ t_2 = -m_2 g \frac{l}{2}\cos q$$

$$t_{ext} = t_1 + t_2 = \frac{gl\cos q\,(m_1 - m_2)}{2}$$

Thus α becomes

$$a = \frac{\sum t_{ext}}{I} = \frac{\frac{1}{2}(m_1 - m_1)gl\ \cos q}{\frac{l^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)} = \frac{2(m_1 - m_1)\cos q}{\left(\frac{1}{3}M + m_1 + m_2\right)} g/l$$

Example 11.8

A start rotates with a period of 30days about an axis through its center. After the star undergoes a supernova explosion, the stellar core, which had a radius of 1.0×10^4km, collapses into a neutron start of radius 3.0km. Determine the period of rotation of the neutron star.

What is your guess about the answer?

The period will be significantly shorter, because its radius got smaller.

Let's make some assumptions:

1. There is no torque acting on it
2. The shape remains spherical
3. Its mass remains constant

Using angular momentum conservation

$$L_i = L_f$$
$$I_i w_i = I_f w_f$$

The angular speed of the star with the period T is

$$w = \frac{2p}{T}$$

Thus

$$w_f = \frac{I_i w_i}{I_f} = \frac{m r_i^2}{m r_f^2}\frac{2p}{T_i}$$

$$T_f = \frac{2p}{w_f} = \left(\frac{r_f^2}{r_i^2}\right) T_i = \left(\frac{3.0}{1.0 \times 10^4}\right)^2 \times 30\, days = 2.7 \times 10^{-6}\, days = 0.23\, s$$

Conditions for Equilibrium

What do you think does the term “An object is at its equilibrium” mean?

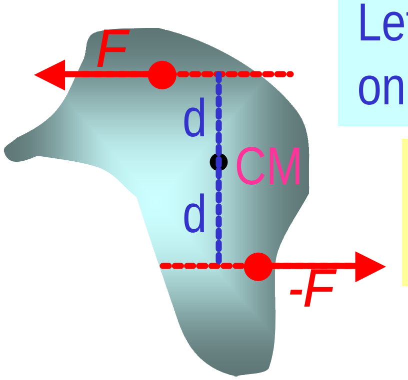
The object is either at rest (<u>Static Equilibrium</u>) or its center of mass is moving with a constant velocity (Dynamic Equilibrium).

When do you think an object is at its equilibrium?

Translational Equilibrium: Equilibrium in linear motion $\sum \vec{F} = 0$

Is this it?

The above condition is sufficient for a point-like particle to be at its static equilibrium. However for object with size this is not sufficient. One more condition is needed. What is it?



Let’s consider two forces equal magnitude but opposite direction acting on a rigid object as shown in the figure. What do you think will happen?

The object will rotate about the CM. The net torque acting on the object about any axis must be 0. $\sum \vec{\tau} = 0$

For an object to be at its static equilibrium, the object should not have linear or angular speed. $v_{CM} = 0$ $\omega = 0$

More on Conditions for Equilibrium

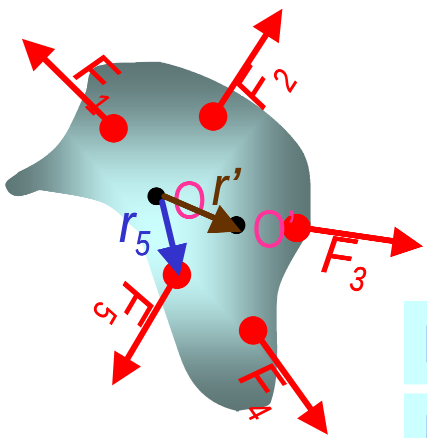
To simplify the problems, we will only deal with forces acting on x-y plane, giving torque only along z-axis. What do you think the conditions for equilibrium be in this case?

The six possible equations from the two vector equations turns to three equations.

$$\sum \vec{F} = 0 \qquad \sum F_x = 0 \qquad \sum F_y = 0$$

$$\sum \vec{\tau} = 0 \qquad \sum \tau_z = 0$$

What happens if there are many forces exerting on the object?



If an object is at its translational static equilibrium, and if the net torque acting on the object is 0 about one axis, the net torque must be 0 about any arbitrary axis.

Net Force exerting on the object

$$\sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots = 0$$

Net torque about O

$$\sum \vec{\tau}_O = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{r}_3 \times \vec{F}_3 + \cdots = \sum \vec{r}_i \times \vec{F}_i = 0$$

Position of force F_i about O'

$$\vec{r}_i{}' = \vec{r}_i - \vec{r}'$$

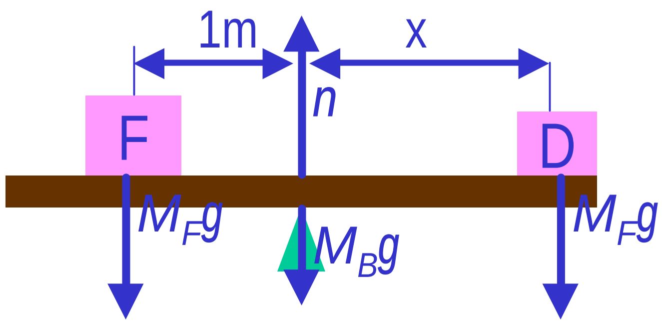
Net torque about O'

$$\sum \vec{\tau}_{O'} = \vec{r}'_1 \times \vec{F}_1 + \vec{r}'_2 \times \vec{F}_2 + \cdots = \left(\vec{r}_1 - \vec{r}'\right) \times \vec{F}_1 + \left(\vec{r}_2 - \vec{r}'\right) \times \vec{F}_2 + \cdots = \sum \vec{r}_i \times \vec{F}_i - \vec{r}' \times \sum \vec{F}_i$$

$$\sum \vec{\tau}_{O'} = \sum \vec{r}_i \times \vec{F}_i - \vec{r}' \times 0 = \sum \vec{\tau}_O = 0$$

Example 12.1

A uniform 40.0 N board supports a father and daughter weighing 800 N and 350 N, respectively. If the support (or fulcrum) is under the center of gravity of the board and the father is 1.00 m from CoG, what is the magnitude of normal force *n* exerted on the board by the support?



Since there is no linear motion, this system is in its translational equilibrium

$$\sum F_x = 0$$

$$\sum F_y = M_B g + M_F g + M_D g - n = 0$$

Therefore the magnitude of the normal force

$$n = 40.0 + 800 + 350 = 1190N$$

Determine where the child should sit to balance the system.

The net torque about the fulcrum by the three forces are

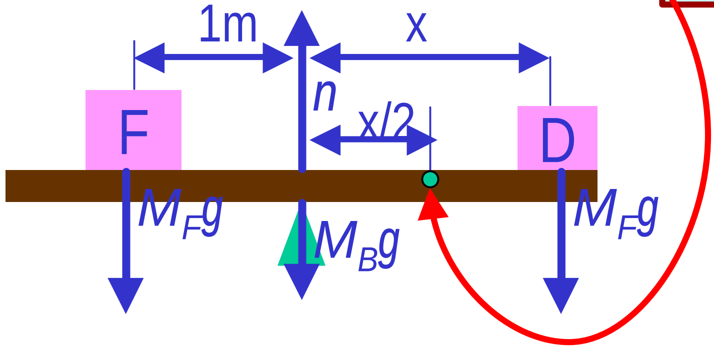
$$t = M_B g \cdot 0 + M_F g \cdot 1.00 - M_D g \cdot x = 0$$

Therefore to balance the system the daughter must sit

$$x = \frac{M_F g}{M_D g} \cdot 1.00m = \frac{800}{350} \cdot 1.00m = 2.29m$$

Example 12.1 Continued

Determine the position of the child to balance the system for different position of axis of rotation.



The net torque about the axis of rotation by all the forces are

$$\tau = M_B g \cdot x/2 + M_F g \cdot (1.00 + x/2) - n \cdot x/2 - M_D g \cdot x/2 = 0$$

Since the normal force is $n = M_B g + M_F g + M_D g$

The net torque can be rewritten

$$\begin{aligned} \tau &= M_B g \cdot x/2 + M_F g \cdot (1.00 + x/2) \\ &- (M_B g + M_F g + M_D g) \cdot x/2 - M_D g \cdot x/2 \\ &= M_F g \cdot 1.00 - M_D g \cdot x = 0 \end{aligned}$$

Therefore

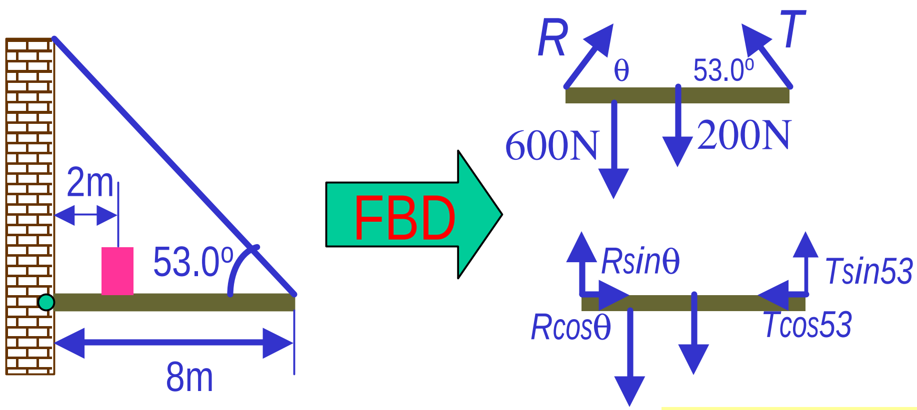
$$x = \frac{M_F g}{M_D g} \cdot 1.00m = \frac{800}{350} \cdot 1.00m = 2.29m$$

What do we learn?

No matter where the rotation axis is, net effect of the torque is identical.

Example 12.3

A uniform horizontal beam with a length of 8.00m and a weight of 200N is attached to a wall by a pin connection. Its far end is supported by a cable that makes an angle of 53.0º with the horizontal. If 600N person stands 2.00m from the wall, find the tension in the cable, as well as the magnitude and direction of the force exerted by the wall on the beam.



First the translational equilibrium, using components

$$\sum F_x = R\cos\theta - T\cos 53.0^\circ = 0$$

$$\sum F_y = R\sin\theta + T\sin 53.0^\circ - 600N - 200N = 0$$

From the rotational equilibrium

$$\sum \tau = T\sin 53.0^\circ \times 8.00 - 600N \times 2.00 - 200N \cdot 4.00m = 0$$

$$T = 313N$$

Using the translational equilibrium

$$R\cos\theta = T\cos 53.0^\circ$$

$$R\sin\theta = -T\sin 53.0^\circ + 600N + 200N$$

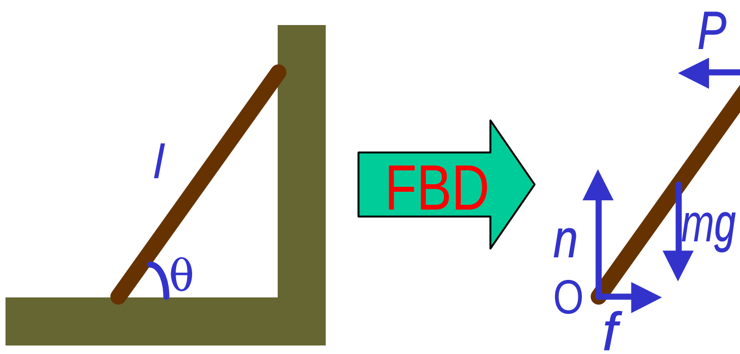
$$\theta = \tan^{-1}\left(\frac{800 - 313\times\sin 53.0^\circ}{313\cos 53.0^\circ}\right) = 71.7^\circ$$

And the magnitude of R is

$$R = \frac{T\cos 53.0^\circ}{\cos\theta} = \frac{313\times\cos 53.0^\circ}{\cos 71.1^\circ} = 582N$$

Example 12.4

A uniform ladder of length *l* and weight *mg*=50 N rests against a smooth, vertical wall. If the coefficient of static friction between the ladder and the ground is μ_s=0.40, find the minimum angle θ_{min} at which the ladder does not slip.



First the translational equilibrium, using components

$$\sum F_x = f - P = 0$$

$$\sum F_y = -mg + n = 0$$

Thus, the normal force is

$$n = mg = 50\,N$$

The maximum static friction force just before slipping is, therefore,

$$f_s^{max} = \mu_s n = 0.4 \times 50\,N = 20\,N = P$$

From the rotational equilibrium

$$\sum \tau_O = -mg\frac{l}{2}\cos\theta_{min} + Pl\sin\theta_{min} = 0$$

$$\theta_{min} = \tan^{-1}\left(\frac{mg}{2P}\right) = \tan^{-1}\left(\frac{50N}{40N}\right) = 51^\circ$$

Example 12.7

A solid brass sphere is initially under normal atmospheric pressure of $1.0\times10^5 N/m^2$. The sphere is lowered into the ocean to a depth at which the pressures is $2.0\times10^7 N/m^2$. The volume of the sphere in air is $0.5m^3$. By how much its volume change once the sphere is submerged?

Since bulk modulus is

$$B = -\frac{\Delta P}{\Delta V / V_i}$$

The amount of volume change is

$$\Delta V = -\frac{\Delta P V_i}{B}$$

From table 12.1, bulk modulus of brass is 6.1×10^{10} N/m^2

The pressure change ΔP is

$$\Delta P = P_f - P_i = 2.0\times10^7 - 1.0\times10^5 \approx 2.0\times10^7$$

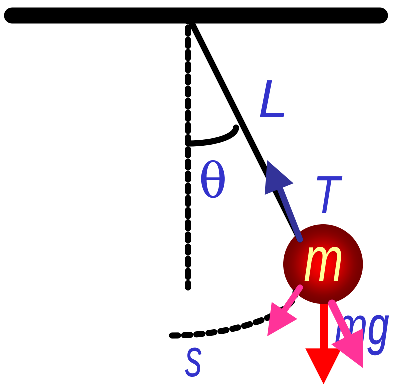
Therefore the resulting volume change ΔV is

$$\Delta V = V_f - V_i = -\frac{2.0\times10^7\times0.5}{6.1\times10^{10}} = -1.6\times10^{-4} m^3$$

The volume has decreased.

The Pendulum

A simple pendulum also performs periodic motion.



The net force exerted on the bob is

$$\sum F_r = T - mg\cos\theta_A = 0$$

$$\sum F_t = -mg\sin\theta_A = ma = m\frac{d^2s}{dt^2}$$

Since the arc length, s, is $s = L\theta_A$

$$\frac{d^2s}{dt^2} = L\frac{d^2\theta}{dt^2} = -g\sin\theta \quad \text{results} \Rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

Again became a second degree differential equation, satisfying conditions for simple harmonic motion

If θ is very small, sinθ~θ

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta = -\omega^2\theta$$

giving angular frequency

$$\omega = \sqrt{\frac{g}{L}}$$

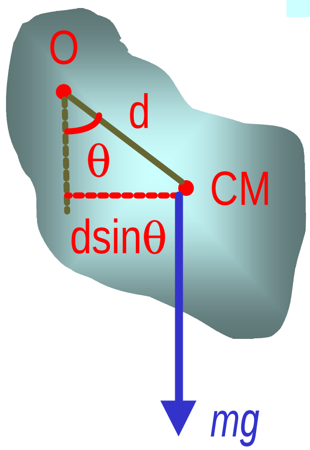
The period for this motion is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

The period only depends on the length of the string and the gravitational acceleration

Physical Pendulum

Physical pendulum is an object that oscillates about a fixed axis which does not go through the object's center of mass.



Consider a rigid body pivoted at a point O that is a distance d from the CM.

The magnitude of the net torque provided by the gravity is

$$\sum t = -mgd \sin q$$

Then

$$\sum t = Ia = I\frac{d^2 q}{dt^2} = -mgd \sin q$$

Therefore, one can rewrite

$$\frac{d^2 q}{dt^2} = -\frac{mgd}{I}\sin q \approx -\left(\frac{mgd}{I}\right)q = -w^2 q$$

Thus, the angular frequency ω is

$$w = \sqrt{\frac{mgd}{I}}$$

And the period for this motion is

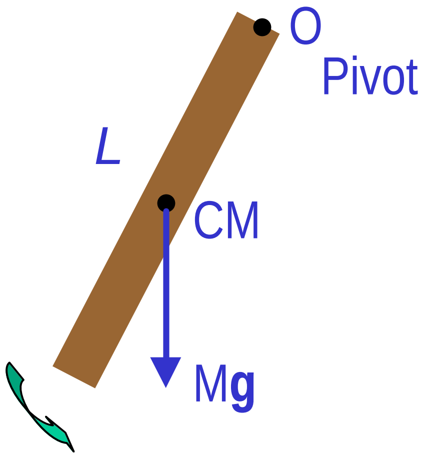
$$T = \frac{2p}{w} = 2p\sqrt{\frac{I}{mgd}}$$

By measuring the period of physical pendulum, one can measure moment of inertia.

Does this work for simple pendulum?

Example 13.6

A uniform rod of mass M and length L is pivoted about one end and oscillates in a vertical plane. Find the period of oscillation if the amplitude of the motion is small.



Moment of inertia of a uniform rod, rotating about the axis at one end is $I = \frac{1}{3}ML^2$

The distance d from the pivot to the CM is L/2, therefore the period of this physical pendulum is

$$T = \frac{2p}{w} = 2p\sqrt{\frac{I}{Mgd}} = 2p\sqrt{\frac{2ML^2}{3MgL}} = 2p\sqrt{\frac{2L}{3g}}$$

Calculate the period of a meter stick that is pivot about one end and is oscillating in a vertical plane.

Since L=1m, the period is

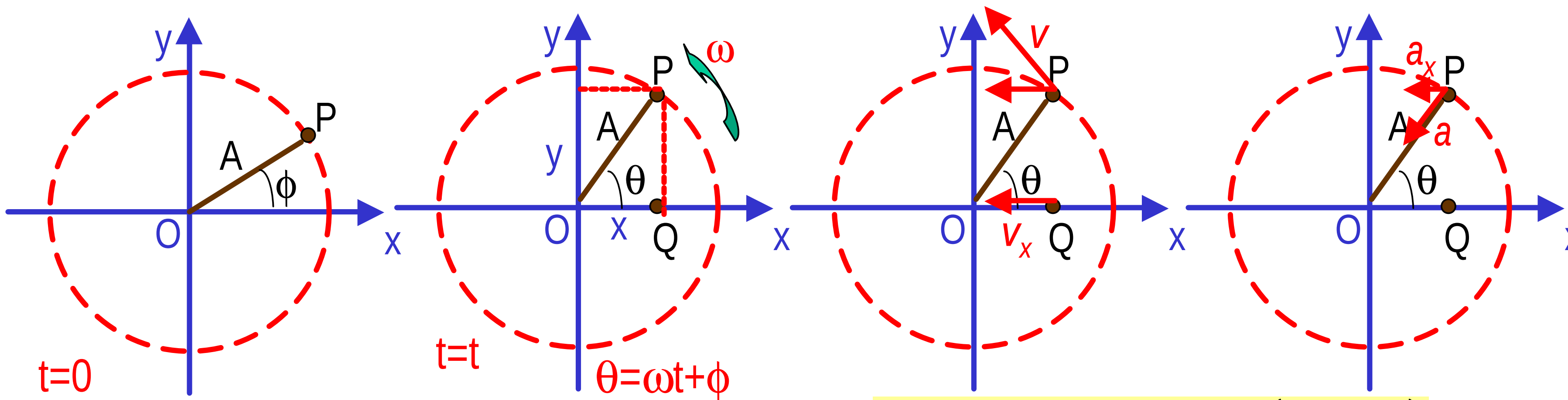
$$T = 2p\sqrt{\frac{2L}{3g}} = 2p\sqrt{\frac{2}{3\cdot 9.8}} = 1.64s$$

So the frequency is

$$f = \frac{1}{T} = 0.61s^{-1}$$

Simple Harmonic and Uniform Circular Motions

Uniform circular motion can be understood as a superposition of two simple harmonic motions in x and y axis.



When the particle rotates at a uniform angular speed ω, x and y coordinate position become

$$x = A\cos\theta = A\cos(\omega t+\phi)$$
$$y = A\sin\theta = A\sin(\omega t+\phi)$$

Since the linear velocity in a uniform circular motion is Aω, the velocity components are

$$v_x = -v\sin\theta = -A\omega\sin(\omega t+\phi)$$
$$v_y = +v\cos\theta = A\omega\cos(\omega t+\phi)$$

Since the radial acceleration in a uniform circular motion is $v^2/A=\omega^2 A$, the components are

$$a_x = -a\cos\theta = -A\omega^2\cos(\omega t+\phi)$$
$$a_y = -a\sin\theta = -A\omega^2\sin(\omega t+\phi)$$

Example 13.7

A particle rotates counterclockwise in a circle of radius 3.00m with a constant angular speed of 8.00 rad/s. At t=0, the particle has an x coordinate of 2.00m and is moving to the right. A) Determine the x coordinate as a function of time.

Since the radius is 3.00m, the amplitude of oscillation in x direction is 3.00m. And the angular frequency is 8.00rad/s. Therefore the equation of motion in x direction is

$$x = A\cos\theta = (3.00m)\cos(8.00t + \phi)$$

Since x=2.00, when t=0

$$2.00 = (3.00m)\cos\phi; \quad \phi = \cos^{-1}\left(\frac{2.00}{3.00}\right) = 48.2^\circ$$

However, since the particle was moving to the right ϕ=-48.2°,

$$x = (3.00m)\cos\left(8.00t - 48.2^\circ\right)$$

Find the x components of the particle's velocity and acceleration at any time t.

Using the displcement

$$v_x = \frac{dx}{dt} = -(3.00\cdot 8.00)\sin(8.00t - 48.2) = (-24.0m/s)\sin\left(8.00t - 48.2^\circ\right)$$

Likewise, from velocity

$$a_x = \frac{dv}{dt} = (-24.0\cdot 8.00)\cos(8.00t - 48.2) = \left(-192m/s^2\right)\cos\left(8.00t - 48.2^\circ\right)$$

Damped Oscillation

More realistic oscillation where an oscillating object loses its mechanical energy in time by a retarding force such as friction or air resistance.

Let's consider a system whose retarding force is air resistance R=-b**v** (b is called damping coefficient) and restoration force is -kx

$$\sum F_x = -kx - bv = ma_x$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

The solution for the above 2nd order differential equation is

$$x = Ae^{-\frac{b}{2m}t}\cos(wt+f)$$

The angular frequency ω for this motion is

$$w = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

This equation of motion tells us that when the retarding force is much smaller than restoration force, the system oscillates but the amplitude decreases, and ultimately, the oscillation stops.

We express the angular frequency as

$$w = \sqrt{w_0^2 - \left(\frac{b}{2m}\right)^2}$$

Where the natural frequency ω_0

$$w_0 = \sqrt{\frac{k}{m}}$$

Free Fall Acceleration & Gravitational Force

Weight of an object with mass *m* is *mg*. Using the force exerting on a particle of mass m on the surface of the Earth, one can get

$$mg = G\frac{M_E m}{R_E^2}$$

$$g = G\frac{M_E}{R_E^2}$$

What would the gravitational acceleration be if the object is at an altitude h above the surface of the Earth?

$$F_g = mg' = G\frac{M_E m}{r} = G\frac{M_E m}{(R_E + h)^2}$$

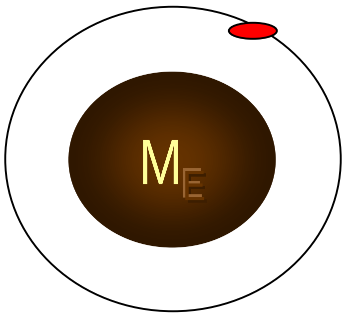
$$g' = G\frac{M_E}{(R_E + h)^2}$$

What do these tell us about the gravitational acceleration?

- The gravitational acceleration is independent of the mass of the object
- The gravitational acceleration decreases as the altitude increases
- If the distance from the surface of the Earth gets infinitely large, the weight of the object approaches 0.

Example 14.2

The international space station is designed to operate at an altitude of 350km. When completed, it will have a weight (measured on the surface of the Earth) of 4.22×10^6N. What is its weight when in its orbit?



The total weight of the station on the surface of the Earth is

$$F_E = mg = G\frac{M_E m}{R_E^{\ 2}} = 4.22 \times 10^6\,N$$

Since the orbit is at 350km above the surface of the Earth, the gravitational force at that height is

$$F_O = mg' = G\frac{M_E m}{\left(R_E + h\right)^2} = \frac{R_E^2}{\left(R_E + h\right)^2} F_E$$

Therefore the weight in the orbit is

$$F_O = \frac{R_E^2}{\left(R_E + h\right)^2} F_E = \frac{\left(6.37 \times 10^6\right)^2}{\left(6.37 \times 10^6 + 3.50 \times 10^5\right)^2} \times 4.22 \times 10^6 = 3.80 \times 10^6\,N$$

Example 14.3

Using the fact that g=9.80m/s^2 at the Earth's surface, find the average density of the Earth.

Since the gravitational acceleration is

$$g = G\frac{M_E}{R_E^{\;2}} = 6.67\times10^{-11}\frac{M_E}{R_E^{\;2}}$$

So the mass of the Earth is

$$M_E = \frac{R_E^{\;2}g}{G}$$

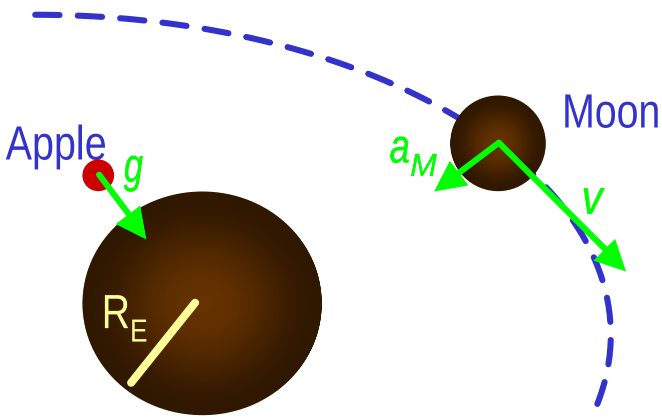
Therefore the density of the Earth is

$$r = \frac{M_E}{V_E} = \frac{\frac{R_E^{\;2}g}{G}}{\frac{4p}{3}R_E^{\;3}} = \frac{3g}{4pGR_E}$$

$$= \frac{3\times9.80}{4p\times6.67\times10^{-11}\times6.37\times10^{6}} = 5.50\times10^{3}kg/m^{3}$$

The Law of Gravity and the Motion of Planets

- Newton assumed that the law of gravitation applies the same whether it is on the Moon or the apple on the surface of the Earth.
- The interacting bodies are assumed to be point like particles.



Newton predicted that the ratio of the Moon's acceleration a_M to the apple's acceleration g would be

$$\frac{a_M}{g} = \frac{(1/r_M)^2}{(1/R_E)^2} = \left(\frac{R_E}{r_M}\right)^2 = \left(\frac{6.37\times10^6}{3.84\times10^8}\right)^2 = 2.75\times10^{-4}$$

Therefore the centripetal acceleration of the Moon, a_M, is

$$a_M = 2.75\times10^{-4}\times9.80 = 2.70\times10^{-3}\,m/s^2$$

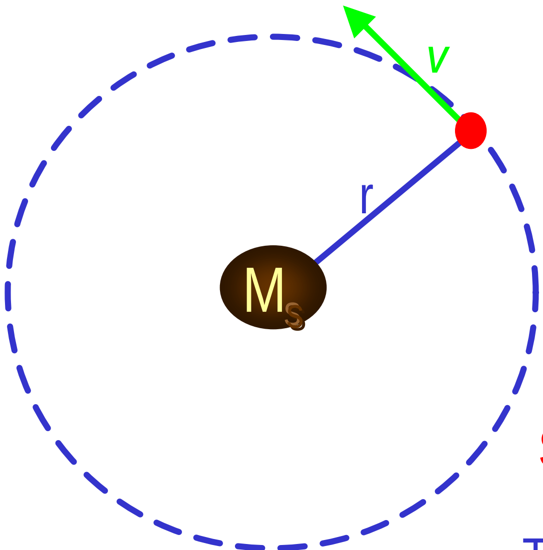
Newton also calculated the Moon's orbital acceleration a_M from the knowledge of its distance from the Earth and its orbital period, T=27.32 days=2.36×10^6s

$$a_M = \frac{v^2}{r_M} = \frac{(2pr_M/T)^2}{r_M} = \frac{4p^2 r_M}{T} = \frac{4p^2\times3.84\times10^8}{2.36\times10^6} = 2.72\times10^{-3}\,m/s^2 \approx \frac{9.80}{(60)^2}$$

This means that the Moon's distance is about 60 times that of the Earth's radius, its acceleration is reduced by the square of the ratio. This proves that the inverse square law is valid.

Kepler's Third Law

It is crucial to show that Keper's third law can be predicted from the inverse square law for circular orbits.



Since the gravitational force exerted by the Sun is radially directed toward the Sun to keep the planet circle, we can apply Newton's second law

$$\frac{GM_sM_P}{r^2} = \frac{M_pv^2}{r}$$

Since the orbital speed, v, of the planet with period T is $v = \frac{2pr}{T}$

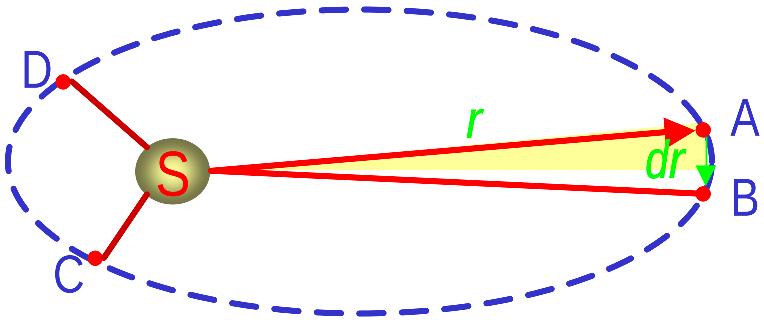
The above can be written $\frac{GM_sM_P}{r^2} = \frac{M_p(2pr/T)^2}{r}$

Solving for T one can obtain $T = \left(\frac{4p^2}{GM_s}\right)r^3 = K_sr^3$ and $K_s = \left(\frac{4p^2}{GM_s}\right) = 2.97\times10^{-19}\, s^2/m^3$

This is Keper's third law. It's also valid for ellipse for r being the length of the semi-major axis. The constant K_s is independent of mass of the planet.

Kepler's Second Law and Angular Momentum Conservation

Consider a planet of mass M_p moving around the Sun in an elliptical orbit.



Since the gravitational force acting on the planet is always toward radial direction, it is a *central force*

Therefore the torque acting on the planet by this force is always 0.

$$\tau = \vec{r} \times \vec{F} = \vec{r} \times F\hat{r} = 0$$

Since torque is the time rate change of angular momentum ***L***, the angular momentum is constant.

$$\tau = \frac{d\vec{L}}{dt} = 0; \quad \vec{L} = const$$

Because the gravitational force exerted on a planet by the Sun results in no torque, the angular momentum L of the planet is constant.

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times M_p \vec{v} = M_p \vec{r} \times \vec{v} = const$$

Since the area swept by the motion of the planet is

$$dA = \frac{1}{2}\left|\vec{r} \times d\vec{r}\right| = \left|\vec{r} \times \vec{v}dt\right| = \frac{L}{2M_p}dt \quad \Rightarrow \quad \frac{dA}{dt} = \frac{L}{2M_p} = const$$

This is Keper's second law which states that the radius vector from the Sun to a planet sweeps our equal areas in equal time intervals.

More on The Gravitational Potential Energy

Since the gravitational force is a radial force, it only performed work while the path was radial direction only. Therefore, the work performed by the gravitational force that depends on the position becomes

$$dW = \vec{F} \cdot d\vec{r} = F(r)dr \xrightarrow{\text{For the whole path}} W = \int_{r_i}^{r_f} F(r)dr$$

Therefore the potential energy is the negative change of work in the path

$$\Delta U = U_f - U_i = -\int_{r_i}^{r_f} F(r)dr$$

Since the Earth's gravitational force is

$$F(r) = -\frac{GM_E m}{r^2}$$

So the potential energy function becomes

$$U_f - U_i = \int_{r_i}^{r_f} \frac{GM_E m}{r^2} dr = -GM_E m \left[\frac{1}{r_f} - \frac{1}{r_i} \right]$$

Since potential energy only matters for differences, by taking the infinite distance as the initial point of the potential energy, we get

$$U = -\frac{GM_E m}{r}$$

For any two particles?

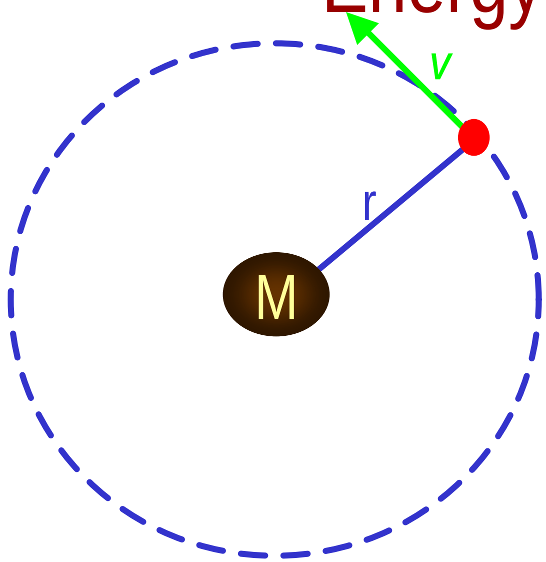
$$U = -\frac{Gm_1 m_2}{r}$$

The energy needed to take the particles infinitely apart.

For many particles?

$$U = \sum_{i,j} U_{i,j}$$

Energy in Planetary and Satellite Motions



Consider an object of mass m moving at a speed *v* near a massive object of mass M (M>>m).

What's the total energy?

$$E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Systems like Sun and Earth or Earth and Moon whose motions are contained within a closed orbit is called *Bound Systems.*

For a system to be bound, the total energy must be negative.

Assuming a circular orbit, in order for the object to be kept in the orbit the gravitational force must provide the radial acceleration. Therefore from Newton's second law of motion

$$\frac{GM_E m}{r^2} = ma = m\frac{v^2}{r}$$

The kinetic energy for this system is

$$\frac{1}{2}mv^2 = \frac{GM_E m}{2r}$$

Therefore the total mechanical energy of the system is

$$E = K + U = -\frac{GMm}{2r}$$

Since the gravitational force is conservative, the total mechanical energy of the system is conserved.

Example 14.7

The space shuttle releases a 470kg communication satellite while in an orbit that is 280km above the surface of the Earth. A rocket engine on the satellite boosts it into a geosynchronous orbit, which is an orbit in which the satellite stays directly over a single location on the Earth, How much energy did the engine have to provide?

What is the radius of the geosynchronous orbit?

$$T = 1day = 8.64 \times 10^{4} s$$

From Kepler's 3rd law $T^{2} = K_{E} r_{GS}^{3}$ Where K_E is $K_{E} = \frac{4p^{2}}{GM_{E}} = 9.89 \times 10^{-14} s^{2} / m^{3}$

Therefore the geosynchronous radius is

$$r_{GS} = \sqrt[3]{\frac{T^{2}}{K_{E}}} = \sqrt[3]{\frac{\left(8.64 \times 10^{4}\right)^{2}}{9.89 \times 10^{-14}}} = \sqrt[3]{\frac{\left(8.64 \times 10^{4}\right)^{2}}{9.89 \times 10^{-14}}} = 4.23 \times 10^{7} m$$

Because the initial position before the boost is 280km

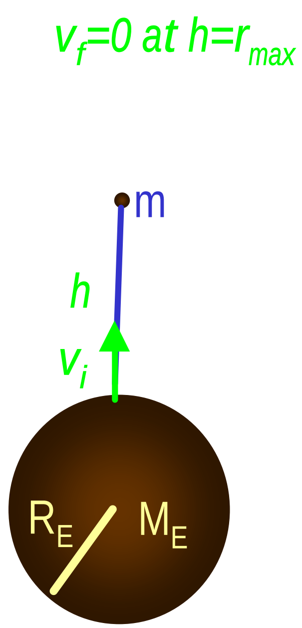
$$r_{i} = R_{E} + 2.80 \times 10^{5} m = 6.65 \times 10^{6} m$$

The total energy needed to boost the satellite at the geosynchronous radius is the difference of the total energy before and after the boost

$$\Delta E = -\frac{GM_{E} m_{s}}{2}\left(\frac{1}{r_{GS}} - \frac{1}{r_{i}}\right)$$

$$= -\frac{6.67\times10^{-11}\times5.98\times10^{24}\times470}{2}\left(\frac{1}{4.23\times10^{7}} - \frac{1}{6.65\times10^{6}}\right) = 1.19\times10^{10} J$$

Escape Speed



Consider an object of mass m is projected vertically from the surface of the Earth with an initial speed v_i and eventually comes to stop $v_f=0$ at the distance r_{max}.

Because the total energy is conserved

$$E = K + U = \frac{1}{2}mv_i^2 - \frac{GM_E m}{R_E} = -\frac{GM_E m}{r_{\max}}$$

Solving the above equation for v_i, one obtains

$$v_i = \sqrt{2GM_E\left(\frac{1}{R_E} - \frac{1}{r_{\max}}\right)}$$

Therefore if the initial speed v_i is known one can use this formula to compute the final height h of the object.

$$h = r_{\max} - R_E = \frac{v_i^2 R_E^2}{2GM_E - v_i^2 R_E}$$

In order for the object to escape Earth's gravitational field completely, the initial speed needs to be

$$v_{esc} = \sqrt{\frac{2GM_E}{R_E}} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{6.37 \times 10^{6}}}$$

$$= 1.12 \times 10^{4}\, m/s = 11.2\, km/s$$

This is called the escape speed. This formula is valid for any planet or large mass objects.

How does this depend on the mass of the escaping object?

Independent of the mass of the escaping object

Fluid and Pressure

What are the three states of matter?

Solid, Liquid, and Gas

How do you distinguish them?

By the time it takes for a particular substance to change its shape in reaction to external forces.

What is a fluid?

A collection of molecules that are randomly arranged and loosely bound by forces between them or by the external container

We will first learn about mechanics of fluid at rest, *fluid statics*.

In what way do you think fluid exerts stress on the object submerged in it?

Fluid cannot exert shearing or tensile stress. Thus, the only force the fluid exerts on an object immersed in it is the forces perpendicular to the surfaces of the object.

This force by the fluid on an object usually is expressed in the form of the force on a unit area at the given depth, the pressure, defined as

$$P \equiv \frac{F}{A}$$

Expression of pressure for an infinitesimal area dA by the force dF is

$$P = \frac{dF}{dA}$$

Note that pressure is a scalar quantity because it's the magnitude of the force on a surface area A.

What is the unit and dimension of pressure?

Unit: N/m^2

Dim.: $[M][L^{-1}][T^{-2}]$

Special SI unit for pressure is Pascal

$$1Pa \equiv 1N/m^2$$

Pascal's Law and Hydraulics

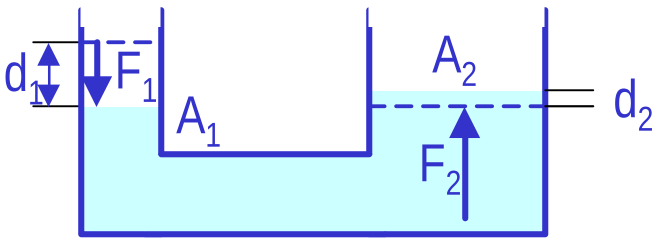
A change in the pressure applied to a fluid is transmitted undiminished to every point of the fluid and to the walls of the container.

$$P = P_0 + rgh$$

What happens if P_0 is changed?

The resultant pressure P at any given depth h increases as much as the change in P_0.

This is the principle behind hydraulic pressure. How?



Since the pressure change caused by the the force F_1 applied on to the area A_1 is transmitted to the F_2 on an area A_2.

$$P = \frac{F_1}{A_1} = \frac{F_2}{A_2}$$

Therefore, the resultant force F_2 is

$$F_2 = \frac{A_2}{A_1} F_1$$

In other words, the force get multiplied by the ratio of the areas A_2/A_1 is transmitted to the F_2 on an area.

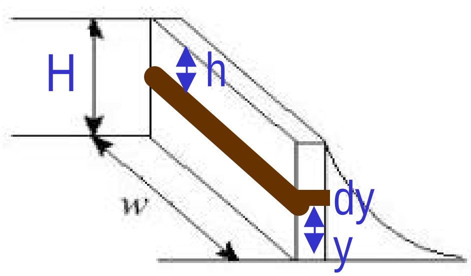
This seems to violate some kind of conservation law, doesn't it?

No, the actual displaced volume of the fluid is the same. And the work done by the forces are still the same.

$$F_2 = \frac{d_1}{d_2} F_1$$

Example 15.4

Water is filled to a height H behind a dam of width w. Determine the resultant force exerted by the water on the dam.



Since the water pressure varies as a function of depth, we will have to do some calculus to figure out the total force.

The pressure at the depth h is

$$P = \rho gh = \rho g(H - y)$$

The infinitesimal force dF exerting on a small strip of dam dy is

$$dF = PdA = \rho g(H - y)wdy$$

Therefore the total force exerted by the water on the dam is

$$F = \int_{y=0}^{y=H} \rho g(H - y)wdy = \rho g\left[Hy - \frac{1}{2}y^2\right]_{y=0}^{y=H} = \frac{1}{2}\rho gH^2$$

Buoyant Forces and Archimedes' Principle

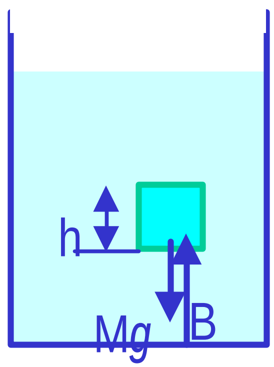
Why is it so hard to put a beach ball under water while a piece of small steel sinks in the water?

The water exerts force on an object immersed in the water.
This force is called Buoyant force.

How does the Buoyant force work?

The magnitude of the buoyant force always equals the weight of the fluid in the volume displaced by the submerged object.

This is called, Archimedes' principle. What does this mean?



Let's consider a cube whose height is h and is filled with fluid and at its equilibrium. Then the weight Mg is balanced by the buoyant force B.

$$B = F_g = Mg$$

And the pressure at the bottom of the cube is larger than the top by ρgh.

Therefore,

$$\Delta P = B / A = \rho g h$$
$$B = \Delta P A = \rho g h A = \rho V g$$
$$B = F_g = \rho V g = Mg$$

Where Mg is the weight of the fluid.

Example 15.5

Archimedes was asked to determine the purity of the gold used in the crown. The legend says that he solved this problem by weighing the crown in air and in water. Suppose the scale read 7.84N in air and 6.86N in water. What should he have to tell the king about the purity of the gold in the crown?

In the air the tension exerted by the scale on the object is the weight of the crown

$$T_{air} = mg = 7.84N$$

In the water the tension exerted by the scale on the object is

$$T_{water} = mg - B = 6.86N$$

Therefore the buoyant force B is

$$B = T_{air} - T_{water} = 0.98N$$

Since the buoyant force B is

$$B = r_w V_w g = r_w V_c g = 0.98N$$

The volume of the displaced water by the crown is

$$V_c = V_w = \frac{0.98N}{r_w g} = \frac{0.98}{1000 \times 9.8} = 1.0 \times 10^{-4} m^3$$

Therefore the density of the crown is

$$r_c = \frac{m_c}{V_c} = \frac{m_c g}{V_c g} = \frac{7.84}{V_c g} = \frac{7.84}{1.0 \times 10^{-4} \times 9.8} = 8.3 \times 10^3 kg/m^3$$

Since the density of pure gold is $19.3 \times 10^3 kg/m^3$, this crown is either not made of pure gold or hollow.

Superposition and Interference

If two or more traveling waves are moving through a medium, the resultant wave function at any point is the algebraic sum of the wave functions of the individual waves.

Superposition Principle

The waves that follow this principle are called ***linear waves*** which in general have small amplitudes. The ones that don't are ***nonlinear waves*** with larger amplitudes.

Thus, one can write the resultant wave function as

$$y = y_1 + y_2 + \cdots + y_n = \sum_{i=1}^{n} y_i$$

Two traveling linear waves can pass through each other without being destroyed or altered.

What do you think will happen to the water waves when you throw two stones on the pond?

They will pass right through each other.

What happens to the waves at the point where they meet?

The shape of wave will change → Interference

Constructive interference: The amplitude increases when the waves meet

Destructive interference: The amplitude decreases when the waves meet

Speed of Waves on Strings

How do we determine the speed of a transverse pulse traveling on a string?

If a string under tension is pulled sideways and released, the tension is responsible for accelerating a particular segment of the string back to the equilibrium position.

So what happens when the tension increases?

The acceleration of the particular segment increases

Which means?

The speed of the wave increases.

Now what happens when the mass per unit length of the string increases?

For the given tension, acceleration decreases, so the wave speed decreases.

Which law does this hypothesis based on?

Newton's second law of motion

Based on the hypothesis we have laid out above, we can construct a hypothetical formula for the speed of wave

$$v = \sqrt{\frac{T}{m}}$$

T: Tension on the string

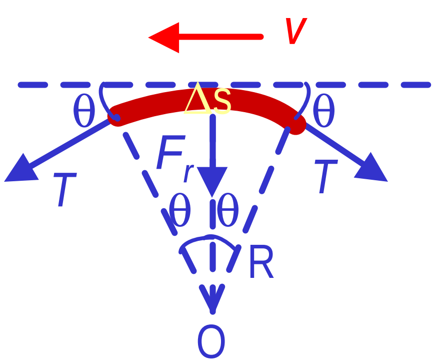
μ: Unit mass per length

Is the above expression dimensionally sound?

$T=[MLT^{-2}]$, $\mu=[ML^{-1}]$

$(T/\mu)^{1/2}=[L^2T^{-2}]^{1/2}=[LT^{-1}]$

Speed of Waves on Strings cont'd



Let's consider a pulse moving to right and look at it in the frame that moves along with the the pulse.

Since in the reference frame moves with the pulse, the segment is moving to the left with the speed *v*, and the centripetal acceleration of the segment is

$$a_r = \frac{v^2}{R}$$

Now what do the force components look in this motion when θ is small?

$$\sum F_t = T\cos\theta - T\cos\theta = 0$$

$$\sum F_r = 2T\sin\theta \approx 2T\theta$$

What is the mass of the segment when the line density of the string is μ?

$$m = \mu\Delta s = \mu R 2\theta = 2\mu R\theta$$

Using the radial force component

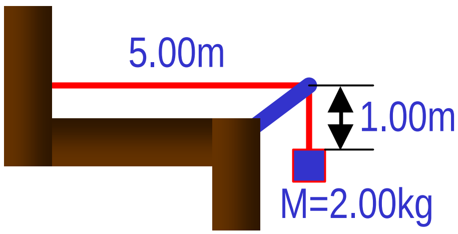
$$\sum F_r = ma = m\frac{v^2}{R} = 2\mu R\theta\frac{v^2}{R} = 2T\theta$$

Therefore the speed of the pulse is

$$v = \sqrt{\frac{T}{\mu}}$$

Example 16.2

A uniform cord has a mass of 0.300kg and a length of 6.00m. The cord passes over a pulley and supports a 2.00kg object. Find the speed of a pulse traveling along this cord.



Since the speed of wave on a string with line density μ and under the tension T is

$$v = \sqrt{\frac{T}{m}}$$

The line density μ is

$$m = \frac{0.300\,kg}{6.00\,m} = 5.00 \times 10^{-2}\,kg\,/\,m$$

The tension on the string is provided by the weight of the object. Therefore

$$T = Mg = 2.00 \times 9.80 = 19.6\,kg \cdot m/s^2$$

Thus the speed of the wave is

$$v = \sqrt{\frac{T}{m}} = \sqrt{\frac{19.6}{5.00 \times 10^{-2}}} = 19.8\,m\,/\,s$$

Reflection and Transmission

A pulse or a wave undergoes various changes when the medium it travels changes.

Depending on how rigid the support is, two radically different reflection patterns can be observed.

1. The support is rigidly fixed: The reflected pulse will be inverted to the original due to the force exerted on to the string by the support in reaction to the force on the support due to the pulse on the string.
2. The support is freely moving: The reflected pulse will maintain the original shape but moving in the reverse direction.

If the boundary is intermediate between the above two extremes, part of the pulse reflects, and the other undergoes transmission, passing through the boundary and propagating in the new medium.

When a wave pulse travels from medium A to B:

- $v_A > v_B$ (or $\mu_A < \mu_B$), the pulse is inverted upon reflection.
- $v_A < v_B$ (or $\mu_A > \mu_B$), the pulse is not inverted upon reflection.

Sinusoidal Waves

Equation of motion of a simple harmonic oscillation is a sine function.

But it does not travel. Now how does wave form look like when the wave travels?

The function describing the position of particles, located at x, of the medium through which the sinusoidal wave is traveling can be written at t=0

$$y = \underbrace{A}_{\text{Amplitude}} \sin\left(\frac{2p}{\underbrace{l}_{\text{Wave Length}}} x\right)$$

The wave form of the wave traveling at the speed v in +x at any given time t becomes

$$y = A\sin\left(\frac{2p}{l}(x - vt)\right)$$

By definition, the speed of wave in terms of wave length and period T is

$$v = \frac{l}{T}$$

Thus the wave form can be rewritten

$$y = A\sin\left[2p\left(\frac{x}{l} - \frac{t}{T}\right)\right]$$

Defining, angular wave number *k* and angular frequency ω,

$$k \equiv \frac{2p}{l}; w = \frac{2p}{T}$$

The wave form becomes

$$y = A\sin(kx - wt)$$

Frequency, *f*,

$$f = \frac{1}{T}$$

Wave speed, *v*

$$v = \frac{l}{T} = \frac{w}{k}$$

General wave form

$$y = A\sin(kx - wt + f)$$

Example 16.3

A sinusoidal wave traveling in the positive x direction has an amplitude of 15.0cm, a wavelength of 40.0cm, and a frequency of 8.00Hz. The vertical displacement of the medium at t=0 and x=0 is also 15.0cm. a) Find the angular wave number k, period T, angular frequency ω, and speed v of the wave.

Using the definition, angular wave number k is

$$k = \frac{2p}{l} = \frac{2p}{0.40} = 5.00p = 15.7 rad / m$$

Period is

$$T = \frac{1}{f} = \frac{1}{8.00} = 0.125 \sec$$

Angular frequency is

$$w = \frac{2p}{T} = 2pf = 50.3 rad / s$$

Using period and wave length, the wave speed is

$$v = \frac{l}{T} = lf = 0.400 \times 8.00 = 3.2 m / s$$

b) Determine the phase constant ϕ, and write a general expression of the wave function.

At x=0 and t=0, y=15.0cm, therefore the phase ϕ becomes

$$y = 0.150 \sin(f) = 0.150$$

$$\sin f = 1; \quad f = \frac{p}{2}$$

Thus the general wave function is

$$y = A\sin(kx - wt + f) = 0.150 \sin\left(15.7x - 50.3t + \frac{p}{2}\right)$$

Sinusoidal Waves on Strings

Let's consider the case where a string is attached to an arm undergoing a simple harmonic oscillation. The trains of waves generated by the motion will travel through the string, causing the particles in the string to undergo simple harmonic motion on y-axis.

If the wave at t=0 is $y = A\sin\left(\frac{2p}{l}x\right)$

What does this mean? $f = 0$

The wave function can be written $y = A\sin(kx - wt)$

This wave function describes the vertical motion of any point on the string at any time t. Therefore, we can use this function to obtain transverse speed, v_y, and acceleration, a_y.

$$v_y = \left.\frac{dy}{dt}\right|_{xconst} = \frac{\partial y}{\partial t} = -wA\cos(kx - wt)$$

$$a_y = \left.\frac{dv_y}{dt}\right|_{xconst} = \frac{\partial v_y}{\partial t} = -w^2 A\sin(kx - wt)$$

These are the speed and acceleration of the particle in the medium not of the wave.

The maximum speed and the acceleration of the particle in the medium at position x at time t are

$$v_{y,\max} = wA$$

$$a_{y,\max} = w^2 A$$

How do these look for simple harmonic motion?

Example 16.4

A string is driven at a frequency of 5.00Hz. The amplitude of the motion is 12.0cm, and the wave speed is 20.0m/s. Determine the angular frequency ω and angular wave number k for this wave, and write and expression for the wave function.

Using frequency, the angular frequency is

$$w = \frac{2p}{\mathrm{T}} = 2pf = 2p \cdot 5.00 = 31.4 rad / s$$

Angular wave number k is

$$k = \frac{2p}{l} = \frac{2p}{vT} = \frac{2pf}{v} = \frac{w}{v} = \frac{31.4}{20.0} = 1.57 rad / m$$

Thus the general expression of the wave function is

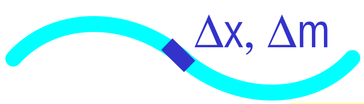
$$y = A\sin(kx - wt) = 0.120\sin(1.57x - 31.4t)$$

Rate of Energy Transfer by Sinusoidal Waves on Strings

Waves traveling through medium carries energy.

When an external source performs work on the string, the energy enters into the string and propagates through the medium as wave.

What is the potential energy of one wave length of a traveling wave?



Elastic potential energy of a particle in a simple harmonic motion $U = \frac{1}{2}ky^2$

Since $\omega^2 = k/m$ $U = \frac{1}{2}mw^2y^2$ The energy ΔU of the segment Δm is $\Delta U = \frac{1}{2}\Delta m w^2 y^2 = \frac{1}{2}m\Delta x w^2 y^2$

As Δx→0, the energy ΔU becomes $dU = \frac{1}{2}mw^2y^2dx$

Using the wave function,the energy is $dU = \frac{1}{2}mw^2A^2\sin^2(kx - wt)dx$

For the wave at t=0, the potential energy in one wave length, λ, is

$$U_l = \frac{1}{2}mw^2A^2\int_{x=0}^{x=l}\sin^2 kx dx = \frac{1}{2}mw^2A^2\int_{x=0}^{x=l}\frac{1-\cos 2kx}{2}dx$$

$$= \frac{1}{2}mw^2A^2\left[\frac{1}{2}x - \frac{1}{4k}\sin\frac{4px}{l}\right]_{x=0}^{x=l} = \frac{1}{4}mw^2A^2 l$$

Recall $k = 2\pi/\lambda$

Rate of Energy Transfer by Sinusoidal Waves cont'd

How does the kinetic energy of each segment of the string in the wave look?

Since the vertical speed of the particle is $v_y = -wA\cos(kx - wt)$

The kinetic energy, ΔK, of the segment Δm is

$$\Delta K = \frac{1}{2}\Delta m v_y^2 = \frac{1}{2}m\Delta x w^2 A^2 \cos^2(kx - wt)$$

As Δx→0, the energy ΔK becomes

$$dK = \frac{1}{2}mw^2 A^2 \cos^2(kx - wt)dx$$

For the wave at t=0, the kinetic energy in one wave length, λ, is

$$K_l = \frac{1}{2}mw^2A^2\int_{x=0}^{x=l}\cos^2 kx dx = \frac{1}{2}mw^2A^2\int_{x=0}^{x=l}\frac{1+\cos 2kx}{2}dx$$

$$= \frac{1}{2}mw^2A^2\left[\frac{1}{2}x + \frac{1}{4k}\sin\frac{4px}{l}\right]_{x=0}^{x=l} = \frac{1}{4}mw^2A^2 l$$

Recall $k=2\pi/\lambda$

Just like harmonic oscillation, the total mechanical energy in one wave length, λ, is

$$E_l = U_l + K_l = \frac{1}{2}mw^2A^2 l$$

As the wave moves along the string, the amount of energy passes by a given point changes during one period. So the power, the rate of energy transfer becomes

$$P = \frac{E_l}{\Delta t} = \frac{1}{2}mw^2A^2\frac{l}{T}$$

$$= \frac{1}{2}mw^2A^2 v$$

P of any sinusoidal wave is proportion to the square of angular frequency, the square of amplitude, density of medium, and wave speed.

Example 16.5

A taut string for which μ=5.00x10^{-2} kg/m is under a tension of 80.0N. How much power must be supplied to the string to generate sinusoidal waves at a frequency of 60.0Hz and an amplitude of 6.00cm?

The speed of the wave is

$$v = \sqrt{\frac{T}{m}} = \sqrt{\frac{80.0}{5.00\times10^{-2}}} = 40.0m/s$$

Using the frequency, angular frequency ω is

$$w = \frac{2p}{\mathrm{T}} = 2pf = 2p \cdot 60.0 = 377\,rad/s$$

Since the rate of energy transfer is

$$P = \frac{E_l}{\Delta t} = \frac{1}{2}mw^2A^2v$$

$$= \frac{1}{2}\times5.00\times10^{-2}\times(377)^2\times(0.06)^2\times(40.0) = 512W$$

Congratulations!!!!

You all have done very well!!!

Good luck with your exams!!!

Have a safe and fun-filled summer!!!