PHYS 5326 – Lecture #14

Monday, Mar. 10, 2003
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- Completion of U(1) Gauge Invariance
- SU(2) Gauge invariance and Yang-Mills Lagrangian

Announcement

- Remember the mid-term exam Friday, Mar. 14, between 10am-noon in room 200
  - Written exam
    - Mostly on concepts to gauge the level of your understanding on the subjects
    - Some simple computations might be necessary
  - Covers up to SU(2) gauge invariance
  - Bring your own pads for the exam
- Review Wednesday, Mar. 12.

U(1) Local Gauge Invariance

The requirement of local gauge invariance forces the introduction of **a massless vector field** into the free Dirac Lagrangian.

$$L = \left[i(\hbar c)\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - (mc^{2})\bar{\psi}\psi\right] + \left[\frac{-1}{16\pi}F^{\mu\nu}F_{\mu\nu}\right] - (q\bar{\psi}\gamma^{\mu}\psi)A_{\mu}$$

Free *L* for gauge field.

Vector field for gauge invariance

A_{μ} is an electromagnetic potential.

And $A_{\mu} \to A_{\mu} + \partial_{\mu}\lambda$ is a gauge transformation of an electromagnetic potential.

U(1) Local Gauge Invariance

The last two terms in Dirac Lagrangian form the Maxwell Lagrangian

$$\mathcal{L}_{Maxwell} = \left[\frac{-1}{16\pi} F^{\mu\nu} F_{\mu\nu}\right] - \frac{1}{c} J^{\mu} A_{\mu}$$

$$= \left[\frac{-1}{16\pi} F^{\mu\nu} F_{\mu\nu}\right] - \left(q\bar{\psi}\gamma^{\mu}\psi\right) A_{\mu}$$

with the current density $J = cq\left(\bar{\psi}\gamma^{\mu}\psi\right)$

U(1) Local Gauge Invariance

Local gauge invariance is preserved if all the derivatives in the lagrangian are replaced by the covariant derivative

$$D_\mu \equiv \partial_\mu + i\frac{q}{\hbar c}A_\mu$$

Minimal Coupling Rule

The gauge transformation preserves local invariance

$$D_\mu \psi \rightarrow \left(\partial_\mu + \frac{iq}{\hbar c}A_\mu\right)e^{-iq\lambda/\hbar c}\psi$$

$$= e^{-iq\lambda/\hbar c}\left[\partial_\mu + \frac{iq}{\hbar c}\left(A_\mu + \partial_\mu \lambda\right)\right]\psi = e^{-iq\lambda/\hbar c}D_\mu\psi$$

Since the gauge transformation, transforms the covariant derivative $D_\mu \rightarrow \partial_\mu + i\frac{q}{\hbar c}\left(A_\mu + \partial_\mu \lambda\right)$

U(1) Gauge Invariance

The global gauge transformation $y \rightarrow e^{iq} y$ is the same as multiplication of ψ by a unitary 1x1 matrix

$$y \rightarrow U y \quad \text{where} \quad U^{+} U = 1 \quad \left(U = e^{iq}\right)$$

The group of all such matrices as U is U(1).

The symmetry involved in gauge transformation is called “U(1) gauge invariance”.

Lagrangian for Two Spin ½ fields

Free Lagrangian for two Dirac fields ψ_1 and ψ_2 with masses m_1 and m_2 is

$$L = \left[i(\hbar c)\bar{\psi}_1 \gamma^\mu \partial_\mu \psi_1 - (m_1 c^2)\bar{\psi}_1 \psi_1 \right] +$$
$$\left[i(\hbar c)\bar{\psi}_2 \gamma^\mu \partial_\mu \psi_2 - (m_2 c^2)\bar{\psi}_2 \psi_2 \right]$$

Applying Euler-Lagrange equation to *L* we obtain Dirac equations for two fields

$$i\gamma^\mu \partial_\mu \psi_1 - \left(\frac{m_1 c}{\hbar}\right)\psi_1 = 0 \qquad i\gamma^\mu \partial_\mu \psi_2 - \left(\frac{m_2 c}{\hbar}\right)\psi_2 = 0$$

Lagrangian for Two Spin ½ fields

By defining a two-component column vector

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Where y_1 and y_2 are four component Dirac spinors

The Lagrangian can be compactified as

$$L = i(\hbar c)\overline{y} g^{m} \partial_{m} y - c^{2}\overline{y} M y$$

With the mass matrix $M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$

Lagrangian for Two Spin ½ fields

If $m_1=m_2$, the Lagrangian looks the same as one particle free Dirac Lagrangian

$$L = i(\hbar c)\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - mc^2\bar{\psi}\psi$$

However, ψ now is a two component column vector.

Global gauge transformation of ψ is $\psi \rightarrow U\psi$.

Where U is any 2x2 unitary matrix $U^{+}U = 1$

Since $\bar{\psi} \rightarrow \bar{\psi}U^{+}$, $\psi\bar{\psi}$ is invariant under the transformation.

SU(2) Gauge Invariance

Any 2x2 unitary matrix can be written, $U = e^{iH}$, where H is a hermitian matrix ($H^{+}=H$).

The matrix H can be generalized by expressing in terms of four real numbers, a_1, a_2, a_3 and q as;

$$H = q\mathbf{1} + \boldsymbol{\tau}\cdot\mathbf{a}$$

Where **1** is the 2x2 unit matrix and $\boldsymbol{\tau}$ is the Pauli matrices

Thus, any unitary 2x2 matrix can be expressed as

$$U = e^{iq\mathbf{1}}e^{i\boldsymbol{\tau}\cdot\mathbf{a}}$$

U(1) gauge

SU(2) gauge

SU(2) Gauge Invariance

The global SU(2) gauge transformation takes the form

$$\psi \rightarrow e^{-i\,\boldsymbol{\tau}\cdot\boldsymbol{\alpha}}\psi$$

Since the determinant of the matrix $e^{-i\,\boldsymbol{\tau}\cdot\boldsymbol{\alpha}}$ is 1, the extended Dirac Lagrangian for two spin ½ fields is invariant under SU(2) global transformations.

Yang and Mills took this global SU(2) invariance to local invariance.

SU(2) Local Gauge Invariance

The local SU(2) gauge transformation by taking the parameter **a** dependent on the position x_μ and defining

$$l \equiv -\frac{\hbar c}{q}\mathbf{a}(x)$$

Where q is a coupling constant analogous to electric charge

is $y \rightarrow Sy$ where $S \equiv e^{-iqt \cdot l(x)/\hbar c}$

L is not invariant under this transformation, since the derivative becomes $\partial_m y \rightarrow S\partial_m y + (\partial_m S)y$

SU(2) Local Gauge Invariance

Local gauge invariance can be preserved by replacing the derivatives with covariant derivative

$$D_m \equiv \partial_m + i \frac{q}{\hbar c} \cdot \mathbf{A}_{\boldsymbol{\mu}}$$

where the vector gauge field follows the transformation rule $D_m y \rightarrow S\left(D_m y\right)$

with a bit more involved manipulation, the resulting *L* that is local gauge invariant is

$$L = i\left(\hbar c\right)\overline{y}\, g^{m} D_m y - mc^2 \overline{y}\, y$$

$$= \left[i\left(\hbar c\right)\overline{y}\, g^{m} \partial_m y - mc^2 \overline{y}\, y\right] - \left(q \overline{y}\, g^{m} y\right) \cdot \mathbf{A}_m$$

SU(2) Local Gauge Invariance

Since the intermediate *L* introduced three new vector fields $A^{\mu}=(A_1^{\mu}, A_2^{\mu}, A_3^{\mu})$, and the *L* requires free *L* for these vector fields

$$L_A=-\frac{1}{16p}F_1^{mn}F_{mn1}-\frac{1}{16p}F_2^{mn}F_{mn2}-\frac{1}{16p}F_3^{mn}F_{mn3}=-\frac{1}{16p}\mathbf{F}_3^{\mu?}\cdot\mathbf{F}_{\mu?3}$$

The Proca mass terms in *L*, $\frac{1}{8p}\left(\frac{mc}{\hbar}\right)^2\mathbf{A}^{?}\cdot\mathbf{A}_{?}=0$ to preserve local gauge invariance, making the vector bosons massless.

This time $F^{mn}=\left(\partial^m A^n-\partial^n A^m\right)$ also does not make the *L* local gauge invariant due to cross terms.

SU(2) Local Gauge Invariance

By redefining

$$\mathbf{F}^{\mu\nu} \equiv \partial^{\mu}\mathbf{A}^{\nu} - \partial^{\nu}\mathbf{A}^{\mu} - \frac{2q}{\hbar c}\left(\mathbf{A}^{\mu} \times \mathbf{A}^{\nu}\right)$$

The complete Yang-Mills Lagrangian *L* becomes

$$L = \left[i(\hbar c)\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - mc^{2}\bar{\psi}\psi\right] - \frac{1}{16\pi}\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu} - \left(q\bar{\psi}\gamma^{\mu}\boldsymbol{\tau}\psi\right)\cdot\mathbf{A}_{\mu}$$

This *L*

- is invariant under SU(2) local gauge transformation.
- describes two equal mass Dirac fields interacting with three massless vector gauge fields.

Yang-Mills Lagrangian

The Dirac fields generates three currents

$$\mathbf{J} \equiv c\left(q\bar{\psi}\gamma^{\mu}\boldsymbol{\tau}\psi\right)$$

These act as sources for the gauge fields whose lagrangian is

$$L_{gauge} = -\frac{1}{16\pi}\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu} - \left(q\bar{\psi}\gamma^{\mu}\boldsymbol{\tau}\psi\right)\cdot\mathbf{A}_{\mu}$$

The complication in SU(2) gauge symmetry stems from the fact that U(2) group is non-Abelian (non-commutative).

Epilogue

Yang-Mills gauge symmetry did not work due to the fact that no-two Dirac particles are equal mass and the requirement of massless iso-triplet vector particle.

This was solved by the introduction of Higgs mechanism to give mass to the vector fields, thereby causing EW symmetry breaking.