PHYS 1441 – Section 004
Lecture #18

Wednesday, Apr. 7, 2004

Dr. Jaehoon Yu

- Torque
- Moment of Inertia
- Rotational Kinetic Energy
- Angular Momentum and its conservation
- Conditions for Equilibrium
- Elasticity

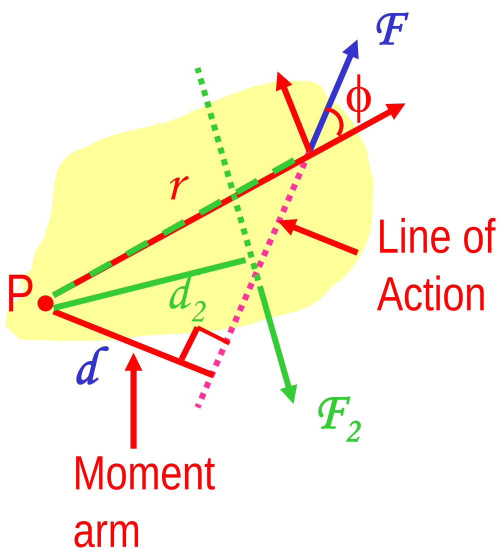
Today's homework is #10 due noon, next Wednesday, Apr. 13, 2004!!

Exam Result and Announcements

- Mid-term grade one-on-one discussion
  - I had only 12 students so far.
  - In my office, RM 242-A, SH
  - During office hours: 2:30 – 3:30 pm, Mondays and Wednesdays
    - Next Monday: Last name starts with A – M
    - Next Wednesday: Last name starts with N – Z

Torque

Torque is the tendency of a force to rotate an object about an axis. Torque, τ, is a vector quantity.



Consider an object pivoting about the point P by the force $\mathcal{F}$ being exerted at a distance r.

The line that extends out of the tail of the force vector is called the line of action.

The perpendicular distance from the pivoting point P to the line of action is called Moment arm.

Magnitude of torque is defined as the product of the force exerted on the object to rotate it and the moment arm.

$$\tau \equiv rF\sin\phi = Fd$$

When there are more than one force being exerted on certain points of the object, one can sum up the torque generated by each force vectorially. The convention for sign of the torque is positive if rotation is in counter-clockwise and negative if clockwise.

$$\sum \tau = \tau_1 + \tau_2$$
$$= F_1 d_1 - F_2 d_2$$

Moment of Inertia

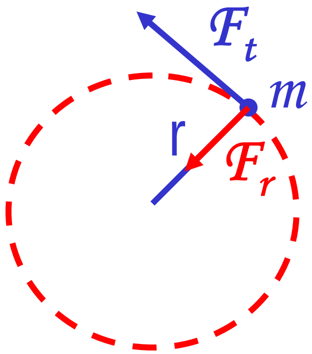
Rotational Inertia: Measure of resistance of an object to changes in its rotational motion. Equivalent to mass in linear motion.

For a group of particles $\boxed{I \equiv \sum_i m_i r_i^2}$ For a rigid body $\boxed{I \equiv \int r^2 dm}$

What are the dimension and unit of Moment of Inertia? $\left[ML^2\right]$ $kg \cdot m^2$

Determining Moment of Inertia is extremely important for computing equilibrium of a rigid body, such as a building.

Torque & Angular Acceleration



Let's consider a point object with mass m rotating on a circle.

What forces do you see in this motion?

The tangential force $\mathcal{F}_t$ and radial force $\mathcal{F}_r$

The tangential force $\mathcal{F}_t$ is $F_t = ma_t = mr\alpha$

The torque due to tangential force $\mathcal{F}_t$ is $\tau = F_t r = ma_t r = mr^2\alpha = I\alpha$

What do you see from the above relationship? $\tau = I\alpha$

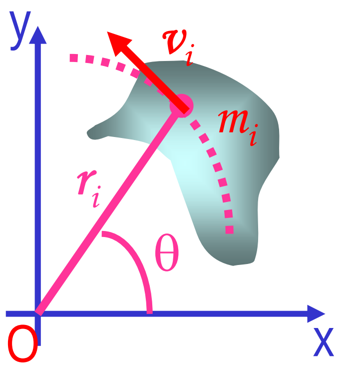
What does this mean?

Torque acting on a particle is proportional to the angular acceleration.

What law do you see from this relationship?

Analogs to Newton's 2nd law of motion in rotation.

Rotational Kinetic Energy



What do you think the kinetic energy of a rigid object that is undergoing a circular motion is?

Kinetic energy of a masslet, m_i, moving at a tangential speed, v_i, is

$$K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 \omega^2$$

Since a rigid body is a collection of masslets, the total kinetic energy of the rigid object is

$$K_R = \sum_i K_i = \frac{1}{2} \sum_i m_i r_i^2 \omega^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

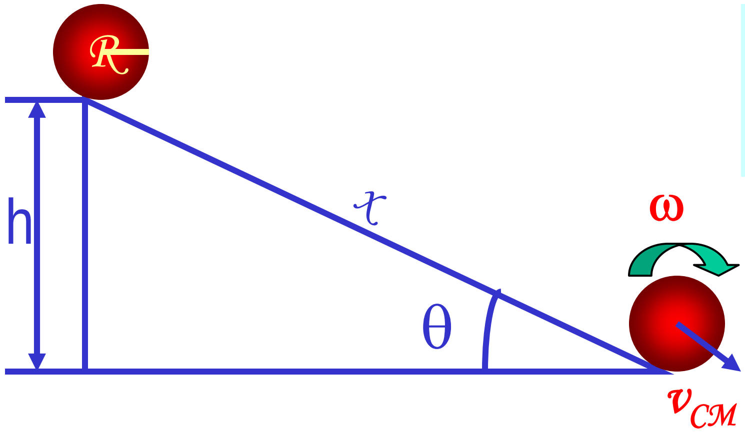
Since moment of Inertia, I, is defined as

$$I = \sum_i m_i r_i^2$$

The above expression is simplified as

$$K_R = \frac{1}{2} I \omega^2$$

Kinetic Energy of a Rolling Sphere



Let's consider a sphere with radius R rolling down a hill without slipping.

$$K = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} MR^2\omega^2$$

$$= \frac{1}{2} I_{CM}\left(\frac{v_{CM}}{R}\right)^2 + \frac{1}{2} M v_{CM}^2$$

$$= \frac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right) v_{CM}^2$$

Since $v_{CM} = R\omega$

What is the speed of the CM in terms of known quantities and how do you find this out?

Since the kinetic energy at the bottom of the hill must be equal to the potential energy at the top of the hill

$$K = \frac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right) v_{CM}^2 = Mgh$$

$$v_{CM} = \sqrt{\frac{2gh}{1 + I_{CM}/MR^2}}$$

Conservation of Angular Momentum

Remember under what condition the linear momentum is conserved?

Linear momentum is conserved when the net external force is 0.

$$\sum \vec{F} = 0 = \frac{d\vec{p}}{dt}$$

$$\vec{p} = const$$

By the same token, the angular momentum of a system is constant in both magnitude and direction, if the resultant external torque acting on the system is 0.

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt} = 0$$

$$\vec{L} = const$$

What does this mean?

Angular momentum of the system before and after a certain change is the same.

$$\vec{L}_i = \vec{L}_f = \text{constant}$$

Three important conservation laws for isolated system that does not get affected by external forces

$$K_i + U_i = K_f + U_f$$ Mechanical Energy

$$\vec{p}_i = \vec{p}_f$$ Linear Momentum

$$\vec{L}_i = \vec{L}_f$$ Angular Momentum

Example for Angular Momentum Conservation

A star rotates with a period of 30days about an axis through its center. After the star undergoes a supernova explosion, the stellar core, which had a radius of 1.0×10^4km, collapses into a neutron start of radius 3.0km. Determine the period of rotation of the neutron star.

What is your guess about the answer?

The period will be significantly shorter, because its radius got smaller.

Let's make some assumptions:

1. There is no torque acting on it
2. The shape remains spherical
3. Its mass remains constant

Using angular momentum conservation

$$L_i = L_f$$

$$I_i \omega_i = I_f \omega_f$$

The angular speed of the star with the period T is

$$\omega = \frac{2\pi}{T}$$

Thus

$$\omega_f = \frac{I_i \omega_i}{I_f} = \frac{m r_i^2}{m r_f^2} \frac{2\pi}{T_i}$$

$$T_f = \frac{2\pi}{\omega_f} = \left(\frac{r_f^2}{r_i^2} \right) T_i = \left(\frac{3.0}{1.0 \times 10^4} \right)^2 \times 30\ days = 2.7 \times 10^{-6}\ days = 0.23\ s$$

Similarity Between Linear and Rotational Motions

All physical quantities in linear and rotational motions show striking similarity.

Quantities	Linear	Rotational
Mass	Mass M	Moment of Inertia $I = mr^2$
Length of motion	Distance L	Angle θ (Radian)
Speed	$v = \frac{\Delta r}{\Delta t}$	$\omega = \frac{\Delta \theta}{\Delta t}$
Acceleration	$a = \frac{\Delta v}{\Delta t}$	$\alpha = \frac{\Delta \omega}{\Delta t}$
Force	Force $\vec{F} = m\vec{a}$	Torque $\vec{\tau} = I\vec{\alpha}$
Work	Work $W = Fd\cos\theta$	Work $W = \tau\theta$
Power	$P = Fv\cos\theta$	$P = \tau\omega$
Momentum	$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{\omega}$
Kinetic Energy	Kinetic $K = \frac{1}{2}mv^2$	Rotational $K_R = \frac{1}{2}I\omega^2$

Conditions for Equilibrium

What do you think does the term "An object is at its equilibrium" mean?

The object is either at rest (Static Equilibrium) or its center of mass is moving with a constant velocity (Dynamic Equilibrium).

When do you think an object is at its equilibrium?

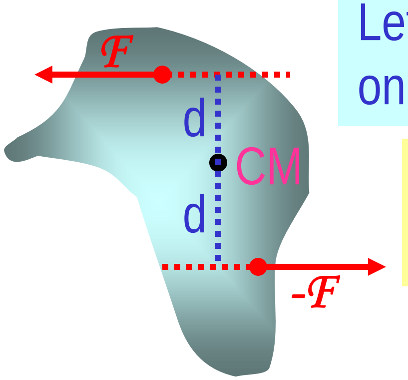
Translational Equilibrium: Equilibrium in linear motion

$$\sum \vec{F} = 0$$

Is this it?

The above condition is sufficient for a point-like particle to be at its static equilibrium. However for object with size this is not sufficient. One more condition is needed. What is it?

Let's consider two forces equal magnitude but opposite direction acting on a rigid object as shown in the figure. What do you think will happen?



The object will rotate about the CM. The net torque acting on the object about any axis must be 0.

$$\sum \vec{\tau} = 0$$

For an object to be at its *static equilibrium*, the object should not have linear or angular speed.

$$v_{CM} = 0 \quad \omega = 0$$

More on Conditions for Equilibrium

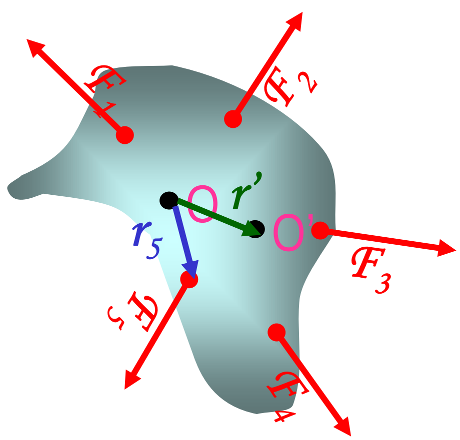
To simplify the problem, we will only deal with forces acting on x-y plane, giving torque only along z-axis. What do you think the conditions for equilibrium be in this case?

The six possible equations from the two vector equations turns to three equations.

$$\sum \vec{F} = 0 \qquad \sum F_x = 0 \qquad \sum F_y = 0$$

$$\sum \vec{\tau} = 0 \qquad \sum \tau_z = 0$$

What happens if there are many forces exerting on the object?



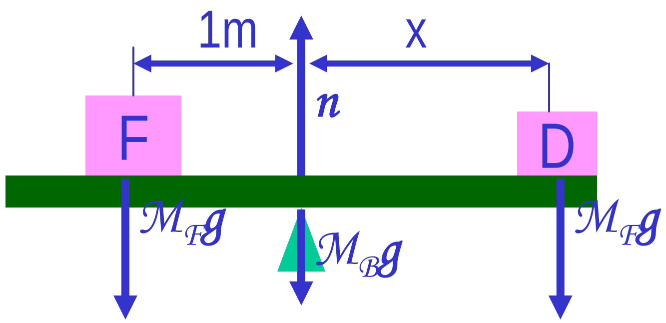
If an object is at its translational static equilibrium, and if the net torque acting on the object is 0 about one axis, the net torque must be 0 about any arbitrary axis.

Why is this true?

Because the object is ***not moving***, no matter what the rotational axis is, there should not be a motion. It is simply a matter of mathematical calculation.

Example for Mechanical Equilibrium

A uniform 40.0 N board supports a father and daughter weighing 800 N and 350 N, respectively. If the support (or fulcrum) is under the center of gravity of the board and the father is 1.00 m from CoG, what is the magnitude of normal force n exerted on the board by the support?



Since there is no linear motion, this system is in its translational equilibrium

$$\sum F_x = 0$$

$$\sum F_y = M_B g + M_F g + M_D g - n = 0$$

Therefore the magnitude of the normal force $n = 40.0 + 800 + 350 = 1190N$

Determine where the child should sit to balance the system.

The net torque about the fulcrum by the three forces are

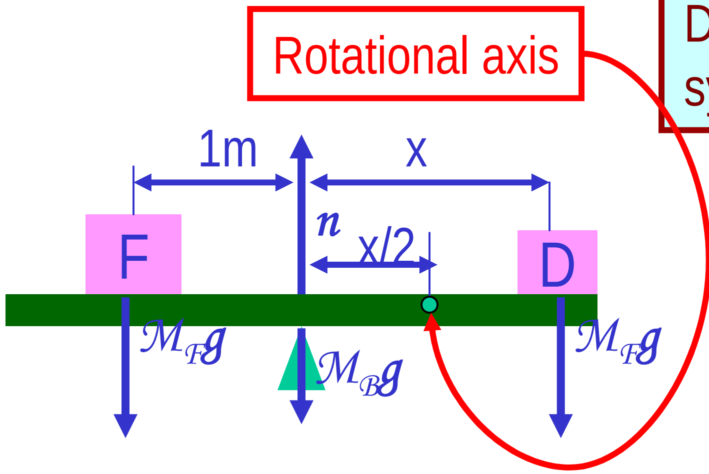
$$\tau = M_B g \cdot 0 + M_F g \cdot 1.00 - M_D g \cdot x = 0$$

Therefore to balance the system the daughter must sit

$$x = \frac{M_F g}{M_D g} \cdot 1.00m = \frac{800}{350} \cdot 1.00m = 2.29m$$

Example for Mech. Equilibrium Cont'd

Determine the position of the child to balance the system for different position of axis of rotation.



The net torque about the axis of rotation by all the forces are

$$\tau = M_B g \cdot x/2 + M_F g \cdot (1.00 + x/2) - n \cdot x/2 - M_D g \cdot x/2 = 0$$

Since the normal force is $n = M_B g + M_F g + M_D g$

The net torque can be rewritten

$$\tau = M_B g \cdot x/2 + M_F g \cdot (1.00 + x/2) - (M_B g + M_F g + M_D g) \cdot x/2 - M_D g \cdot x/2$$
$$= M_F g \cdot 1.00 - M_D g \cdot x = 0$$

Therefore

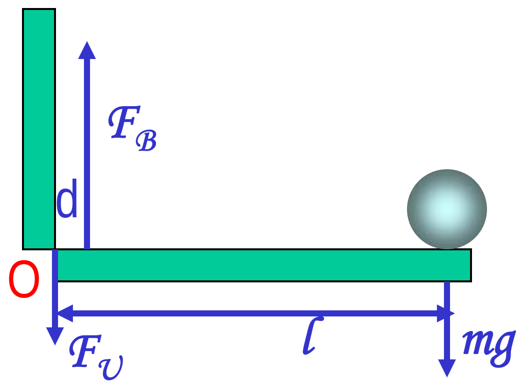
$$x = \frac{M_F g}{M_D g} \cdot 1.00m = \frac{800}{350} \cdot 1.00m = 2.29m$$

What do we learn?

No matter where the rotation axis is, net effect of the torque is identical.

Example for Mechanical Equilibrium

A person holds a 50.0N sphere in his hand. The forearm is horizontal. The biceps muscle is attached 3.00 cm from the joint, and the sphere is 35.0cm from the joint. Find the upward force exerted by the biceps on the forearm and the downward force exerted by the upper arm on the forearm and acting at the joint. Neglect the weight of forearm.



Since the system is in equilibrium, from the translational equilibrium condition

$$\sum F_x = 0$$

$$\sum F_y = F_B - F_U - mg = 0$$

From the rotational equilibrium condition $\sum \tau = F_U \cdot 0 + F_B \cdot d - mg \cdot l = 0$

Thus, the force exerted by the biceps muscle is

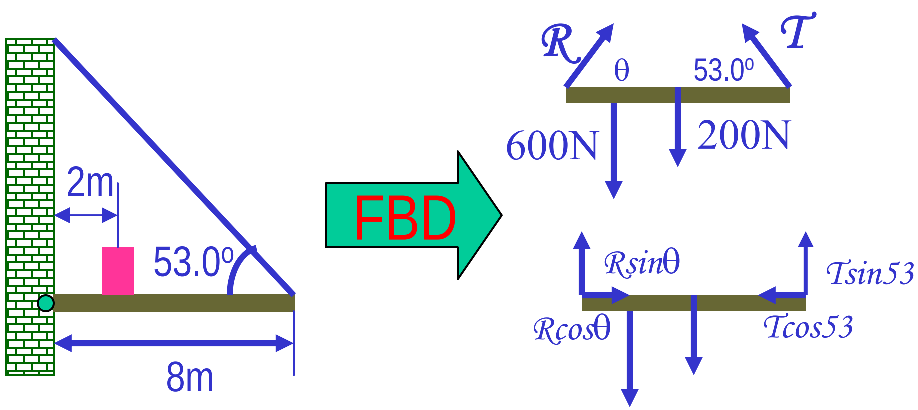
$$F_B \cdot d = mg \cdot l$$

$$F_B = \frac{mg \cdot l}{d} = \frac{50.0 \times 35.0}{3.00} = 583N$$

Force exerted by the upper arm is $F_U = F_B - mg = 583 - 50.0 = 533\,N$

Example for Mechanical Equilibrium

A uniform horizontal beam with a length of 8.00m and a weight of 200N is attached to a wall by a pin connection. Its far end is supported by a cable that makes an angle of 53.0º with the horizontal. If 600N person stands 2.00m from the wall, find the tension in the cable, as well as the magnitude and direction of the force exerted by the wall on the beam.



First the translational equilibrium, using components

$$\sum F_x = R\cos\theta - T\cos 53.0^\circ = 0$$

$$\sum F_y = R\sin\theta + T\sin 53.0^\circ - 600N - 200N = 0$$

From the rotational equilibrium

$$\sum \tau = T\sin 53.0^\circ \times 8.00 - 600N \times 2.00 - 200N \cdot 4.00m = 0$$

$$T = 313N$$

Using the translational equilibrium

$$R\cos\theta = T\cos 53.0^\circ$$

$$R\sin\theta = -T\sin 53.0^\circ + 600N + 200N$$

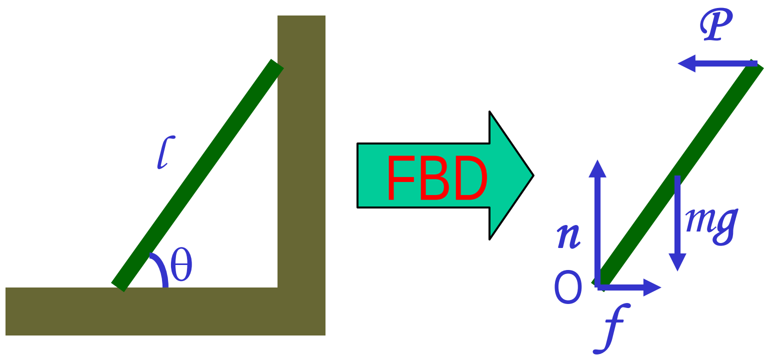
$$\theta = \tan^{-1}\left(\frac{800 - 313\times\sin 53.0^\circ}{313\cos 53.0^\circ}\right) = 71.7^\circ$$

And the magnitude of R is

$$R = \frac{T\cos 53.0^\circ}{\cos\theta} = \frac{313\times\cos 53.0^\circ}{\cos 71.1^\circ} = 582N$$

Example for Mechanical Equilibrium

A uniform ladder of length ℓ and weight mg=50 N rests against a smooth, vertical wall. If the coefficient of static friction between the ladder and the ground is μ_s=0.40, find the minimum angle θ_{min} at which the ladder does not slip.



First the translational equilibrium, using components

$$\sum F_x = f - P = 0$$

$$\sum F_y = -mg + n = 0$$

Thus, the normal force is

$$n = mg = 50\,N$$

The maximum static friction force just before slipping is, therefore,

$$f_s^{\max} = \mu_s n = 0.4 \times 50N = 20N = P$$

From the rotational equilibrium

$$\sum \tau_O = -mg\frac{l}{2}\cos\theta_{\min} + Pl\sin\theta_{\min} = 0$$

$$\theta_{\min} = \tan^{-1}\left(\frac{mg}{2P}\right) = \tan^{-1}\left(\frac{50N}{40N}\right) = 51^{\circ}$$

How did we solve equilibrium problems?

1. Identify all the forces and their directions and locations
2. Draw a free-body diagram with forces indicated on it
3. Write down vector force equation for each x and y component with proper signs
4. Select a rotational axis for torque calculations → Selecting the axis such that the torque of one of the unknown forces become 0.
5. Write down torque equation with proper signs
6. Solve the equations for unknown quantities