PHYS 1441 – Section 004
Lecture #25

Wednesday, May 5, 2004
Dr. Jaehoon Yu

Review of Chapters 8 - 11

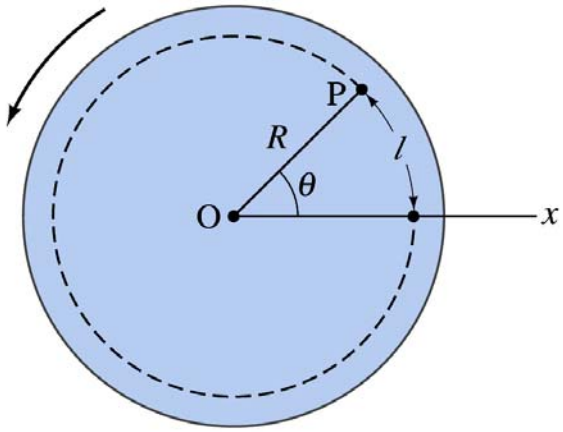
Final Exam at 11am – 12:30pm, Next Monday, May. 10 in SH101!

Fundamentals on Rotation

Linear motions can be described as the motion of the center of mass with all the mass of the object concentrated on it.

Is this still true for rotational motions?

No, because different parts of the object have different linear velocities and accelerations.



Consider a motion of a rigid body – an object that does not change its shape – rotating about the axis protruding out of the slide.

The arc length, or sergita, is $l = R\theta$

Therefore the angle, θ, is $\theta = \frac{l}{R}$. *And the unit of the angle is in radian.*

One radian is the angle swept by an arc length equal to the radius of the arc.

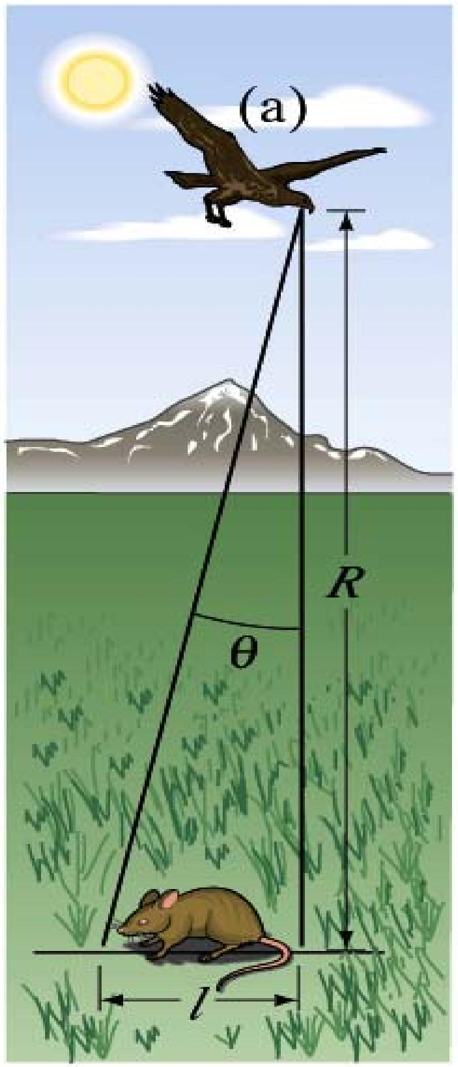
Since the circumference of a circle is 2πr, $360^\circ = 2\pi r / r = 2\pi$

The relationship between radian and degrees is $1\ \text{rad} = 360^\circ / 2\pi = 180^\circ / \pi$

$\cong 180^\circ / 3.14 \cong 57.3^\circ$

Example 8-1

A particular bird's eyes can just distinguish objects that subtend an angle no smaller than about 3x10-4 rad. (a) How many degrees is this? (b) How small an object can the bird just distinguish when flying at a height of 100m?



(a) One radian is 360º/2π. Thus

$$3\times10^{-4}\,rad = \left(3\times10^{-4}\,rad\right)\times\left(360^\circ/2\pi\,rad\right) = 0.017^\circ$$

(b) Since l=rθ and for small angle arc length is approximately the same as the chord length.

$$l = r\theta = 100m\times3\times10^{-4}\,rad = 3\times10^{-2}\,m = 3cm$$

Angular Displacement, Velocity, and Acceleration

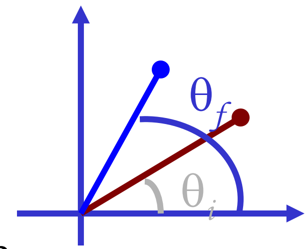
Using what we have learned in the previous slide, how would you define the angular displacement?

$$\Delta\theta = \theta_f - \theta_i$$

How about the average angular speed?

Unit? rad/s

$$\overline{\omega} \equiv \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$



And the instantaneous angular speed?

Unit? rad/s

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

By the same token, the average angular acceleration

Unit? rad/s^2

$$\overline{\alpha} \equiv \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

And the instantaneous angular acceleration? Unit? rad/s^2

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

When rotating about a fixed axis, every particle on a rigid object rotates through the same angle and has the same angular speed and angular acceleration.

Rotational Kinematics

The first type of motion we have learned in linear kinematics was under a constant acceleration. We will learn about the rotational motion under constant angular acceleration about a fixed rotational axis, because these are the simplest motions in both cases.

Just like the case in linear motion, one can obtain

Angular Speed under constant angular acceleration:

$$\omega_f = \omega_i + \alpha t$$

Angular displacement under constant angular acceleration:

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$$

One can also obtain

$$\omega_f^2 = \omega_i^2 + 2\alpha\left(\theta_f - \theta_i\right)$$

Example for Rotational Kinematics

A wheel rotates with a constant angular acceleration of 3.50 rad/s^2. If the angular speed of the wheel is 2.00 rad/s at t_i=0, a) through what angle does the wheel rotate in 2.00s?

Using the angular displacement formula in the previous slide, one gets

$$\theta_f - \theta_i = \omega t + \frac{1}{2}\alpha t^2$$

$$= 2.00 \times 2.00 + \frac{1}{2} 3.50 \times (2.00)^2 = 11.0\, rad$$

$$= \frac{11.0}{2\pi} rev. = 1.75 rev.$$

Example for Rotational Kinematics cnt'd

What is the angular speed at t=2.00s?

Using the angular speed and acceleration relationship

$$\omega_f = \omega_i + \alpha t \quad = 2.00 + 3.50 \times 2.00 = 9.00 rad/s$$

Find the angle through which the wheel rotates between t=2.00 s and t=3.00 s.

Using the angular kinematic formula

$$\theta_f - \theta_i \quad = \omega t + \frac{1}{2}\alpha t^2$$

At t=2.00s

$$\theta_{t=2} = \quad 2.00 \times 2.00 + \frac{1}{2} 3.50 \times 2.00 \quad = 11.0 rad$$

At t=3.00s

$$\theta_{t=3} = 2.00 \times 3.00 + \frac{1}{2} 3.50 \times (3.00)^2 \quad = 21.8 rad$$

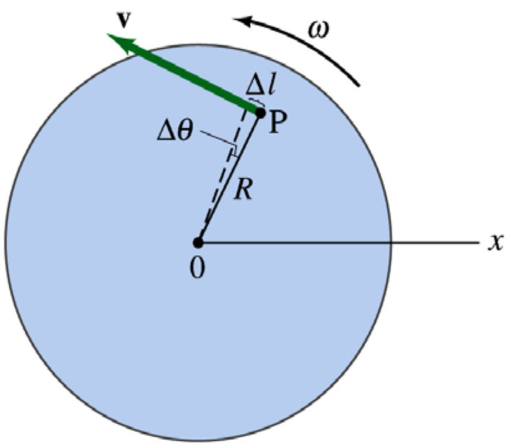
Angular displacement

$$\Delta\theta \quad = \theta_3 - \theta_2 = 10.8 rad \quad = \frac{10.8}{2\pi} rev. = 1.72 rev.$$

Relationship Between Angular and Linear Quantities

What do we know about a rigid object that rotates about a fixed axis of rotation?

Every particle (or masslet) in the object moves in a circle centered at the axis of rotation.



When a point rotates, it has both the linear and angular motion components in its motion.

What is the linear component of the motion you see?

Linear velocity along the tangential direction.

How do we related this linear component of the motion with angular component?

The direction of ω follows a right-hand rule.

The arc-length is $\boxed{l = R\theta}$ So the tangential speed v is $v = \frac{\Delta l}{\Delta t} = \frac{\Delta}{\Delta t}(r\theta) = r\frac{\Delta\theta}{\Delta t} = r\omega$

What does this relationship tell you about the tangential speed of the points in the object and their angular speed?:

Although every particle in the object has the same angular speed, its tangential speed differs proportional to its distance from the axis of rotation.

The farther away the particle is from the center of rotation, the higher the tangential speed.

Example 8-3

(a) What is the linear speed of a child seated 1.2m from the center of a steadily rotating merry-go-around that makes one complete revolution in 4.0s? (b) What is her total linear acceleration?

First, figure out what the angular speed of the merry-go-around is.

$$\varpi = \frac{1rev}{4.0s} = \frac{2\pi rad}{4.0s} = 1.6rad/s$$

Using the formula for linear speed

$$v = r\omega = 1.2m \times 1.6rad/s = 1.9m/s$$

Since the angular speed is constant, there is no angular acceleration.

Tangential acceleration is

$$a_t = r\alpha = 1.2m \times 0rad/s^2 = 0m/s^2$$

Radial acceleration is

$$a_r = r\varpi^2 = 1.2m \times (1.6rad/s)^2 = 3.1m/s^2$$

Thus the total acceleration is

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{0 + (3.1)^2} = 3.1m/s^2$$

Rolling Motion of a Rigid Body

What is a rolling motion?

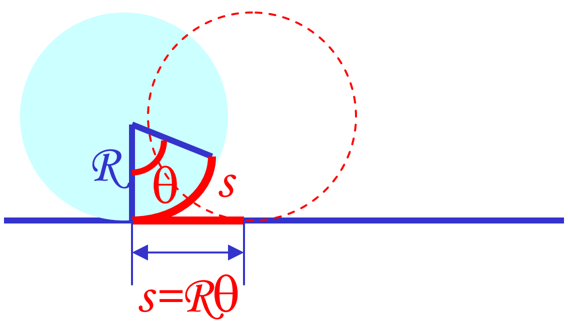
A more generalized case of a motion where the rotational axis moves together with the object

A rotational motion about the moving axis

To simplify the discussion, let's make a few assumptions

1. Limit our discussion on very symmetric objects, such as cylinders, spheres, etc
2. The object rolls on a flat surface

Let's consider a cylinder rolling without slipping on a flat surface



Under what condition does this "Pure Rolling" happen?

The total linear distance the CM of the cylinder moved is

$$s = R\theta$$

Thus the linear speed of the CM is

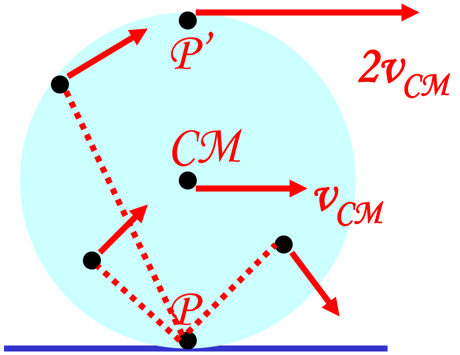
$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

Condition for "Pure Rolling"

More Rolling Motion of a Rigid Body

The magnitude of the linear acceleration of the CM is

$$a_{CM} = \frac{\Delta v_{CM}}{\Delta t} = R\frac{\Delta \omega}{\Delta t} = R\alpha$$



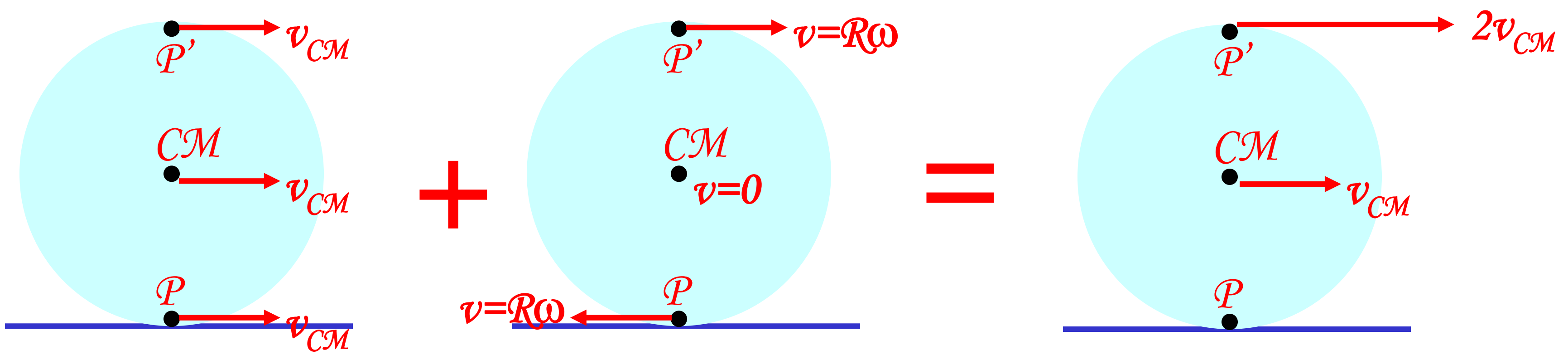
As we learned in the rotational motion, all points in a rigid body moves at the same angular speed but at a different linear speed.

At any given time the point that comes to P has 0 linear speed while the point at P' has twice the speed of CM

Why??

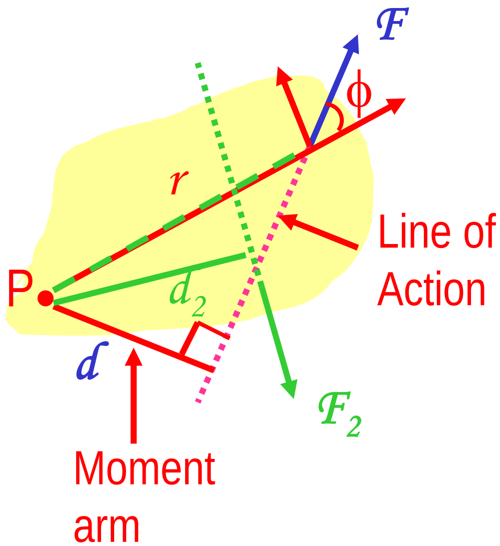
CM is moving at the same speed at all times.

A rolling motion can be interpreted as the sum of Translation and Rotation



Torque

Torque is the tendency of a force to rotate an object about an axis. Torque, τ, is a vector quantity.



Consider an object pivoting about the point P by the force F being exerted at a distance r.

The line that extends out of the tail of the force vector is called the line of action.

The perpendicular distance from the pivoting point P to the line of action is called Moment arm.

Magnitude of torque is defined as the product of the force exerted on the object to rotate it and the moment arm.

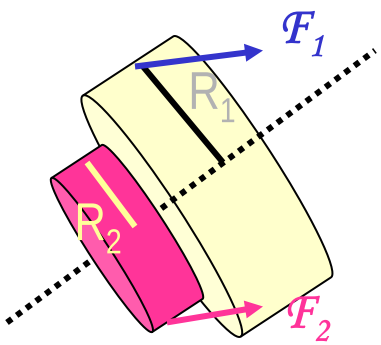
$$\tau \equiv rF\sin\phi = Fd$$

When there are more than one force being exerted on certain points of the object, one can sum up the torque generated by each force vectorially. The convention for sign of the torque is positive if rotation is in counter-clockwise and negative if clockwise.

$$\sum \tau = \tau_1 + \tau_2$$
$$= F_1 d_1 - F_2 d_2$$

Example for Torque

A one piece cylinder is shaped as in the figure with core section protruding from the larger drum. The cylinder is free to rotate around the central axis shown in the picture. A rope wrapped around the drum whose radius is R_1 exerts force F_1 to the right on the cylinder, and another force exerts F_2 on the core whose radius is R_2 downward on the cylinder. A) What is the net torque acting on the cylinder about the rotation axis?



The torque due to F_1 $\tau_1 = -R_1 F_1$ and due to F_2 $\tau_2 = R_2 F_2$

So the total torque acting on the system by the forces is

$$\sum \tau = \tau_1 + \tau_2 = -R_1 F_1 + R_2 F_2$$

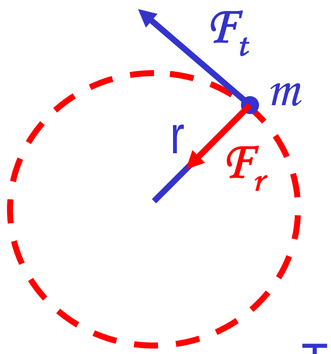
Suppose F_1=5.0 N, R_1=1.0 m, F_2= 15.0 N, and R_2=0.50 m. What is the net torque about the rotation axis and which way does the cylinder rotate from the rest?

Using the above result

$$\sum \tau = -R_1 F_1 + R_2 F_2$$
$$= -5.0 \times 1.0 + 15.0 \times 0.50 = 2.5 N \bullet m$$

The cylinder rotates in counter-clockwise.

Torque & Angular Acceleration



Let's consider a point object with mass m rotating on a circle.

What forces do you see in this motion?

The tangential force F_t and radial force F_r

The tangential force F_t is $$F_t = ma_t = mr\alpha$$

The torque due to tangential force F_t is $$\tau = F_t r = ma_t r = mr^2\alpha = I\alpha$$

What do you see from the above relationship? $$\tau = I\alpha$$

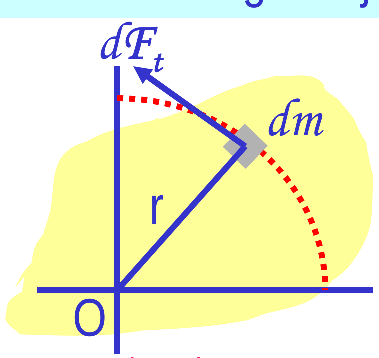
What does this mean?

Torque acting on a particle is proportional to the angular acceleration.

What law do you see from this relationship?

Analogs to Newton's 2nd law of motion in rotation.

How about a rigid object?



The external tangential force dF_t is $$dF_t = dma_t = dmr\alpha$$

The torque due to tangential force F_t is $$d\tau = dF_t r = \left(r^2 dm\right)\alpha$$

The total torque is $$\sum\tau = \alpha\int r^2 dm = I\alpha$$

What is the contribution due to radial force and why?

Contribution from radial force is 0, because its line of action passes through the pivoting point, making the moment arm 0.

Moment of Inertia

Rotational Inertia: Measure of resistance of an object to changes in its rotational motion. Equivalent to mass in linear motion.

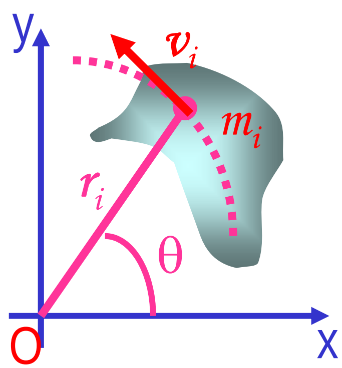
For a group of particles $$I \equiv \sum_i m_i r_i^2$$

For a rigid body $$I \equiv \int r^2 dm$$

What are the dimension and unit of Moment of Inertia? $\left[ML^2\right]$ $kg \cdot m^2$

Determining Moment of Inertia is extremely important for computing equilibrium of a rigid body, such as a building.

Rotational Kinetic Energy



What do you think the kinetic energy of a rigid object that is undergoing a circular motion is?

Kinetic energy of a masslet, m_i, moving at a tangential speed, v_i, is

$$K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 \omega^2$$

Since a rigid body is a collection of masslets, the total kinetic energy of the rigid object is

$$K_R = \sum_i K_i = \frac{1}{2} \sum_i m_i r_i^2 \omega^2 = \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2$$

Since moment of Inertia, I, is defined as

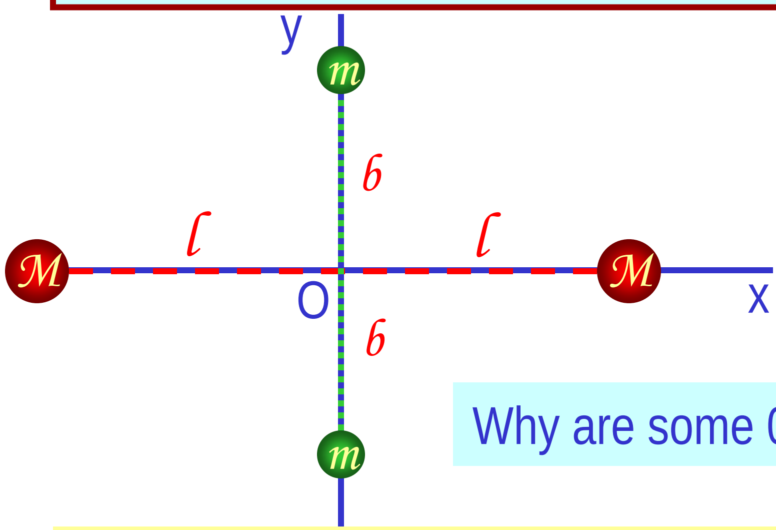
$$I = \sum_i m_i r_i^2$$

The above expression is simplified as

$$K_R = \frac{1}{2} I \omega^2$$

Example for Moment of Inertia

In a system consists of four small spheres as shown in the figure, assuming the radii are negligible and the rods connecting the particles are massless, compute the moment of inertia and the rotational kinetic energy when the system rotates about the y-axis at ω.



Since the rotation is about y axis, the moment of inertia about y axis, I_y, is

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + m\cdot 0^2 + m\cdot 0^2 = 2Ml^2$$

Why are some 0s?

This is because the rotation is done about y axis, and the radii of the spheres are negligible.

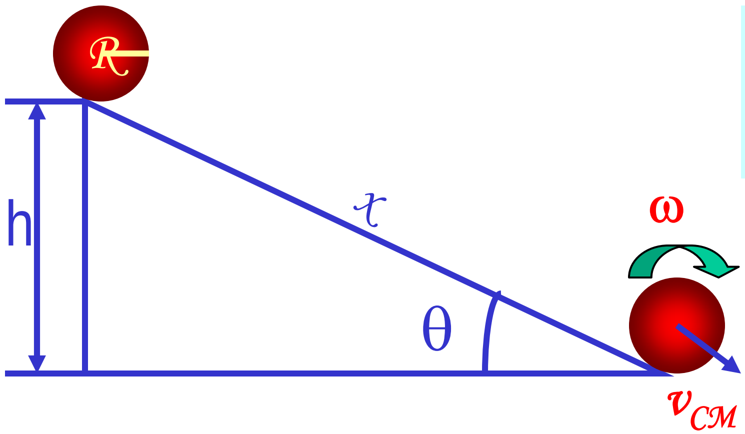
Thus, the rotational kinetic energy is

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2\right)\omega^2 = Ml^2\omega^2$$

Find the moment of inertia and rotational kinetic energy when the system rotates on the x-y plane about the z-axis that goes through the origin O.

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + mb^2 + mb^2 = 2\left(Ml^2 + mb^2\right) \qquad K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2 + 2mb^2\right)\omega^2 = \left(Ml^2 + mb^2\right)\omega^2$$

Kinetic Energy of a Rolling Sphere



Let's consider a sphere with radius R rolling down a hill without slipping.

$$K = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M R^2 \omega^2$$

$$= \frac{1}{2} I_{CM} \left(\frac{v_{CM}}{R} \right)^2 + \frac{1}{2} M v_{CM}^2$$

$$= \frac{1}{2} \left(\frac{I_{CM}}{R^2} + M \right) v_{CM}^2$$

Since $v_{CM} = R\omega$

What is the speed of the CM in terms of known quantities and how do you find this out?

Since the kinetic energy at the bottom of the hill must be equal to the potential energy at the top of the hill

$$K = \frac{1}{2} \left(\frac{I_{CM}}{R^2} + M \right) v_{CM}^2 = Mgh$$

$$v_{CM} = \sqrt{\frac{2gh}{1 + I_{CM} / MR^2}}$$

Conservation of Angular Momentum

Remember under what condition the linear momentum is conserved?

Linear momentum is conserved when the net external force is 0.

$$\sum \vec{F} = 0 = \frac{d\vec{p}}{dt}$$

$$\vec{p} = const$$

By the same token, the angular momentum of a system is constant in both magnitude and direction, if the resultant external torque acting on the system is 0.

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt} = 0$$

$$\vec{L} = const$$

What does this mean?

Angular momentum of the system before and after a certain change is the same.

$$\vec{L}_i = \vec{L}_f = \text{constant}$$

Three important conservation laws for isolated system that does not get affected by external forces

	$K_i + U_i = K_f + U_f$	Mechanical Energy
	$\vec{p}_i = \vec{p}_f$	Linear Momentum
	$\vec{L}_i = \vec{L}_f$	Angular Momentum

Similarity Between Linear and Rotational Motions

All physical quantities in linear and rotational motions show striking similarity.

Quantities	Linear	Rotational
Mass	Mass M	Moment of Inertia $I = mr^2$
Length of motion	Distance L	Angle θ (Radian)
Speed	$v = \frac{\Delta r}{\Delta t}$	$\omega = \frac{\Delta \theta}{\Delta t}$
Acceleration	$a = \frac{\Delta v}{\Delta t}$	$\alpha = \frac{\Delta \omega}{\Delta t}$
Force	Force $\vec{F} = m\vec{a}$	Torque $\vec{\tau} = I\vec{\alpha}$
Work	Work $W = Fd\cos\theta$	Work $W = \tau\theta$
Power	$P = Fv\cos\theta$	$P = \tau\omega$
Momentum	$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{\omega}$
Kinetic Energy	Kinetic $K = \frac{1}{2}mv^2$	Rotational $K_R = \frac{1}{2}I\omega^2$

Conditions for Equilibrium

What do you think does the term "An object is at its equilibrium" mean?

The object is either at rest (Static Equilibrium) or its center of mass is moving with a constant velocity (Dynamic Equilibrium).

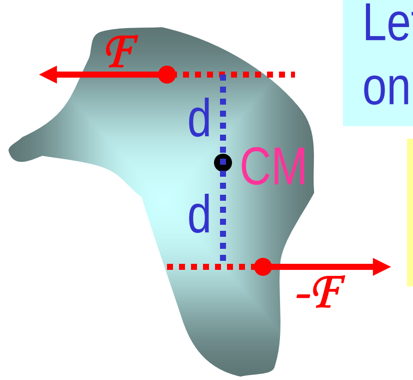
When do you think an object is at its equilibrium?

Translational Equilibrium: Equilibrium in linear motion

$$\sum \vec{F} = 0$$

Is this it?

The above condition is sufficient for a point-like particle to be at its static equilibrium. However for object with size this is not sufficient. One more condition is needed. What is it?



Let's consider two forces equal magnitude but opposite direction acting on a rigid object as shown in the figure. What do you think will happen?

The object will rotate about the CM. The net torque acting on the object about any axis must be 0.

$$\sum \vec{\tau} = 0$$

For an object to be at its *static equilibrium*, the object should not have linear or angular speed.

$$v_{CM} = 0 \quad \omega = 0$$

More on Conditions for Equilibrium

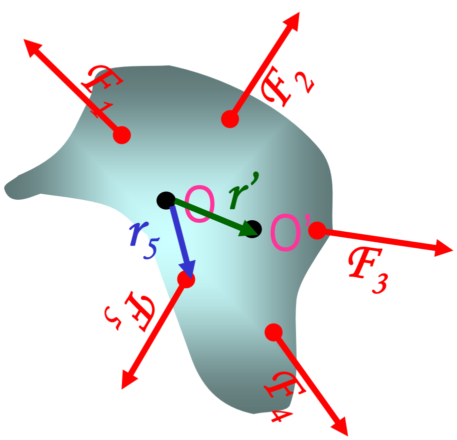
To simplify the problem, we will only deal with forces acting on x-y plane, giving torque only along z-axis. What do you think the conditions for equilibrium be in this case?

The six possible equations from the two vector equations turns to three equations.

$$\sum \vec{F} = 0 \qquad \sum F_x = 0 \qquad\qquad \sum \vec{\tau} = 0 \qquad \sum \tau_z = 0$$

$$\sum F_y = 0$$

What happens if there are many forces exerting on the object?



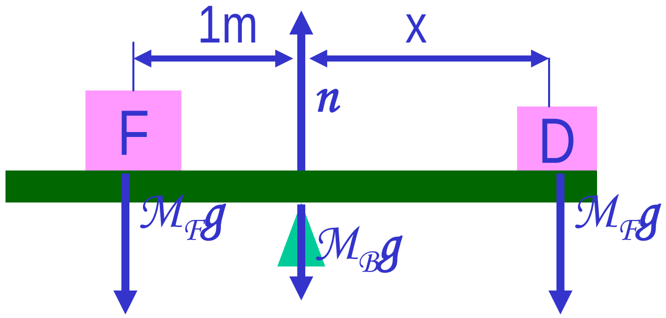
If an object is at its translational static equilibrium, and if the net torque acting on the object is 0 about one axis, the net torque must be 0 about any arbitrary axis.

Why is this true?

Because the object is ***not moving***, no matter what the rotational axis is, there should not be a motion. It is simply a matter of mathematical calculation.

Example for Mechanical Equilibrium

A uniform 40.0 N board supports a father and daughter weighing 800 N and 350 N, respectively. If the support (or fulcrum) is under the center of gravity of the board and the father is 1.00 m from CoG, what is the magnitude of normal force n exerted on the board by the support?



Since there is no linear motion, this system is in its translational equilibrium

$$\sum F_x = 0$$

$$\sum F_y = M_B g + M_F g + M_D g - n = 0$$

Therefore the magnitude of the normal force

$$n = 40.0 + 800 + 350 = 1190 N$$

Determine where the child should sit to balance the system.

The net torque about the fulcrum by the three forces are

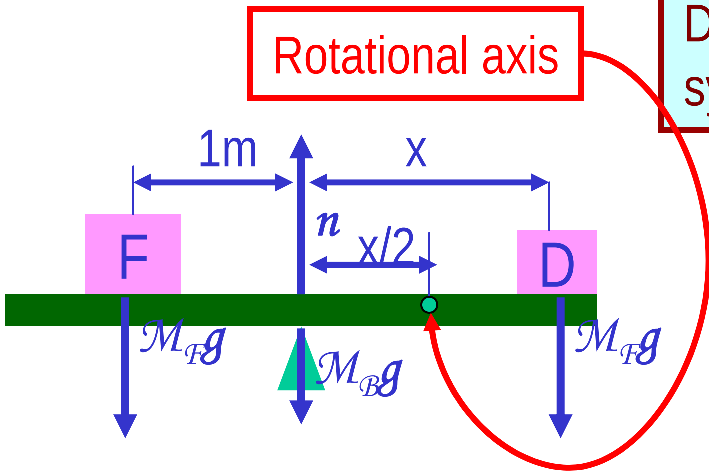
$$\tau = M_B g \cdot 0 + M_F g \cdot 1.00 - M_D g \cdot x = 0$$

Therefore to balance the system the daughter must sit

$$x = \frac{M_F g}{M_D g} \cdot 1.00 m = \frac{800}{350} \cdot 1.00 m = 2.29 m$$

Example for Mech. Equilibrium Cont'd

Determine the position of the child to balance the system for different position of axis of rotation.



The net torque about the axis of rotation by all the forces are

$$\tau = M_B g \cdot x/2 + M_F g \cdot (1.00 + x/2) - n \cdot x/2 - M_D g \cdot x/2 = 0$$

Since the normal force is $n = M_B g + M_F g + M_D g$

The net torque can be rewritten

$$\begin{aligned}\tau &= M_B g \cdot x/2 + M_F g \cdot (1.00 + x/2) \\ &\quad - (M_B g + M_F g + M_D g) \cdot x/2 - M_D g \cdot x/2 \\ &= M_F g \cdot 1.00 - M_D g \cdot x = 0\end{aligned}$$

Therefore

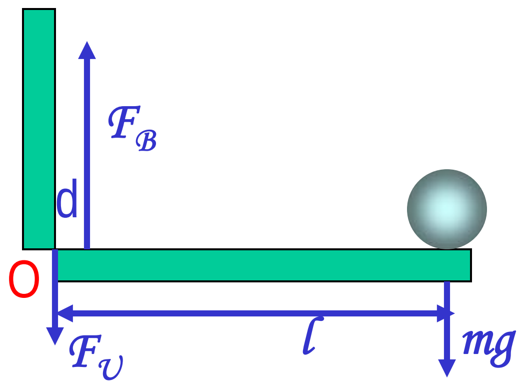
$$x = \frac{M_F g}{M_D g} \cdot 1.00m = \frac{800}{350} \cdot 1.00m = 2.29m$$

What do we learn?

No matter where the rotation axis is, net effect of the torque is identical.

Example for Mechanical Equilibrium

A person holds a 50.0N sphere in his hand. The forearm is horizontal. The biceps muscle is attached 3.00 cm from the joint, and the sphere is 35.0cm from the joint. Find the upward force exerted by the biceps on the forearm and the downward force exerted by the upper arm on the forearm and acting at the joint. Neglect the weight of forearm.



Since the system is in equilibrium, from the translational equilibrium condition

$$\sum F_x = 0$$

$$\sum F_y = F_B - F_U - mg = 0$$

From the rotational equilibrium condition $\sum \tau = F_U \cdot 0 + F_B \cdot d - mg \cdot l = 0$

Thus, the force exerted by the biceps muscle is

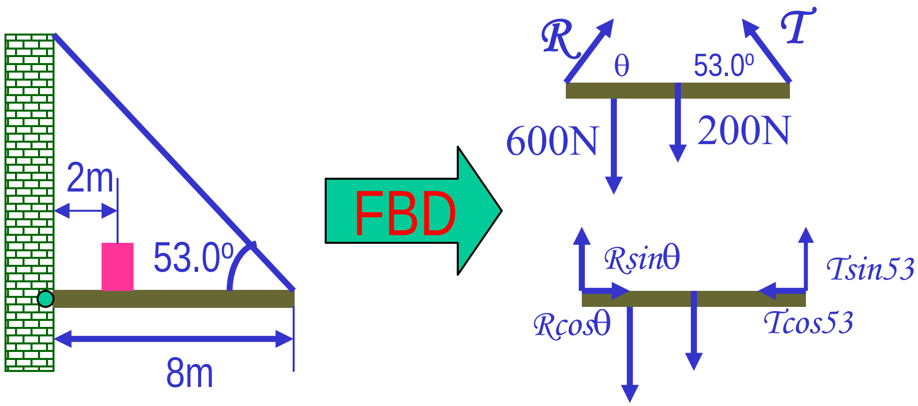
$$F_B \cdot d = mg \cdot l$$

$$F_B = \frac{mg \cdot l}{d} = \frac{50.0 \times 35.0}{3.00} = 583N$$

Force exerted by the upper arm is $F_U = F_B - mg = 583 - 50.0 = 533\,N$

Example for Mechanical Equilibrium

A uniform horizontal beam with a length of 8.00m and a weight of 200N is attached to a wall by a pin connection. Its far end is supported by a cable that makes an angle of 53.0° with the horizontal. If 600N person stands 2.00m from the wall, find the tension in the cable, as well as the magnitude and direction of the force exerted by the wall on the beam.



First the translational equilibrium, using components

$$\sum F_x = R\cos\theta - T\cos 53.0^\circ = 0$$

$$\sum F_y = R\sin\theta + T\sin 53.0^\circ - 600N - 200N = 0$$

From the rotational equilibrium

$$\sum \tau = T\sin 53.0^\circ \times 8.00 - 600N \times 2.00 - 200N \cdot 4.00m = 0$$

$$T = 313N$$

Using the translational equilibrium

$$R\cos\theta = T\cos 53.0^\circ$$

$$R\sin\theta = -T\sin 53.0^\circ + 600N + 200N$$

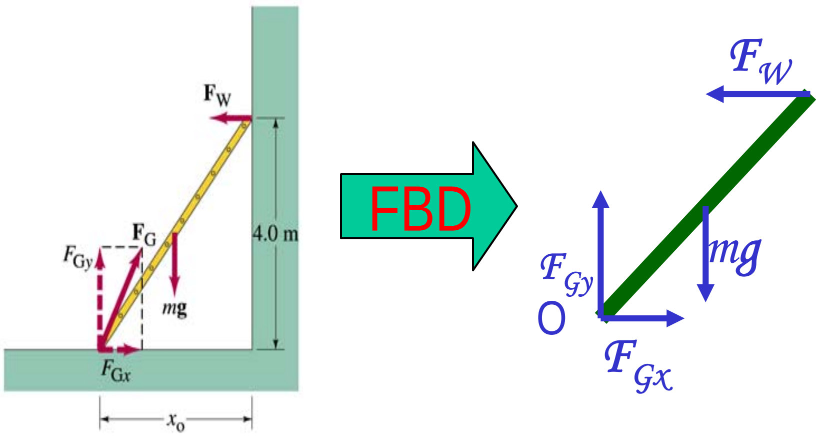
$$\theta = \tan^{-1}\left(\frac{800 - 313\times\sin 53.0^\circ}{313\cos 53.0^\circ}\right) = 71.7^\circ$$

And the magnitude of R is

$$R = \frac{T\cos 53.0^\circ}{\cos\theta} = \frac{313\times\cos 53.0^\circ}{\cos 71.1^\circ} = 582N$$

Example 9 – 9

A 5.0 m long ladder leans against a wall at a point 4.0m above the ground. The ladder is uniform and has mass 12.0kg. Assuming the wall is frictionless (but ground is not), determine the forces exerted on the ladder by the ground and the wall.



First the translational equilibrium, using components

$$\sum F_x = F_{Gx} - F_W = 0$$

$$\sum F_y = -mg + F_{Gy} = 0$$

Thus, the y component of the force by the ground is

$$F_{Gy} = mg = 12.0 \times 9.8N = 118N$$

The length x_0 is, from Pythagorian theorem

$$x_0 = \sqrt{5.0^2 - 4.0^2} = 3.0m$$

Example 9 – 9 cont'd

From the rotational equilibrium $\sum \tau_O = -mg\,x_0/2 + F_W 4.0 \;= 0$

Thus the force exerted on the ladder by the wall is

$$F_W = \frac{mg\,x_0/2}{4.0} = \frac{118 \cdot 1.5}{4.0} = 44N$$

Tx component of the force by the ground is

$$\sum F_x = F_{Gx} - F_W = 0$$

Solve for F_{Gx}

$$F_{Gx} = F_W = 44N$$

Thus the force exerted on the ladder by the ground is

$$F_G = \sqrt{F_{Gx}^2 + F_{Gy}^2} = \sqrt{44^2 + 118^2} \approx 130N$$

The angle between the ladder and the wall is $\theta = \tan^{-1}\left(\frac{F_{Gy}}{F_{Gx}}\right) = \tan^{-1}\left(\frac{118}{44}\right) = 70^\circ$

How do we solve equilibrium problems?

1. Identify all the forces and their directions and locations
2. Draw a free-body diagram with forces indicated on it
3. Write down vector force equation for each x and y component with proper signs
4. Select a rotational axis for torque calculations → Selecting the axis such that the torque from as many of the unknown forces become 0.
5. Write down torque equation with proper signs
6. Solve the equations for unknown quantities

Elastic Properties of Solids

We have been assuming that the objects do not change their shapes when external forces are exerting on it. It this realistic?

No. In reality, the objects get deformed as external forces act on it, though the internal forces resist the deformation as it takes place.

Deformation of solids can be understood in terms of Stress and Strain

Stress: A quantity proportional to the force causing deformation. (Ultimate strength of a material)

Strain: Measure of degree of deformation

It is empirically known that for small stresses, strain is proportional to stress

The constants of proportionality are called Elastic Modulus

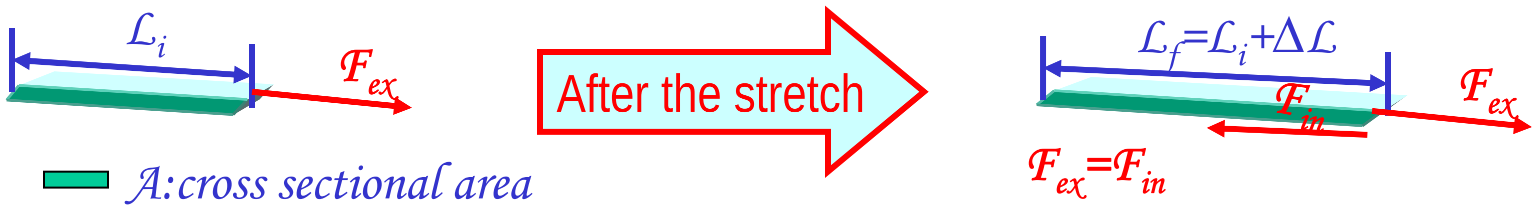
$$\text{Elastic Modulus} \equiv \frac{\text{stress}}{\text{strain}}$$

Three types of Elastic Modulus

1. **Young's modulus**: Measure of the elasticity in length
2. **Shear modulus**: Measure of the elasticity in plane
3. **Bulk modulus**: Measure of the elasticity in volume

Young's Modulus

Let's consider a long bar with cross sectional area A and initial length $\mathcal{L}_i$.



Tensile stress $\text{Tensile Stress} \equiv \frac{F_{ex}}{A}$ Tensile strain $\text{Tensile Strain} \equiv \frac{\Delta L}{L_i}$

Young's Modulus is defined as

$$Y \equiv \frac{\text{Tensile Stress}}{\text{Tensile Strain}} = \frac{F_{ex}/A}{\Delta L/L_i}$$

Used to characterize a rod or wire stressed under tension or compression

What is the unit of Young's Modulus?

Force per unit area

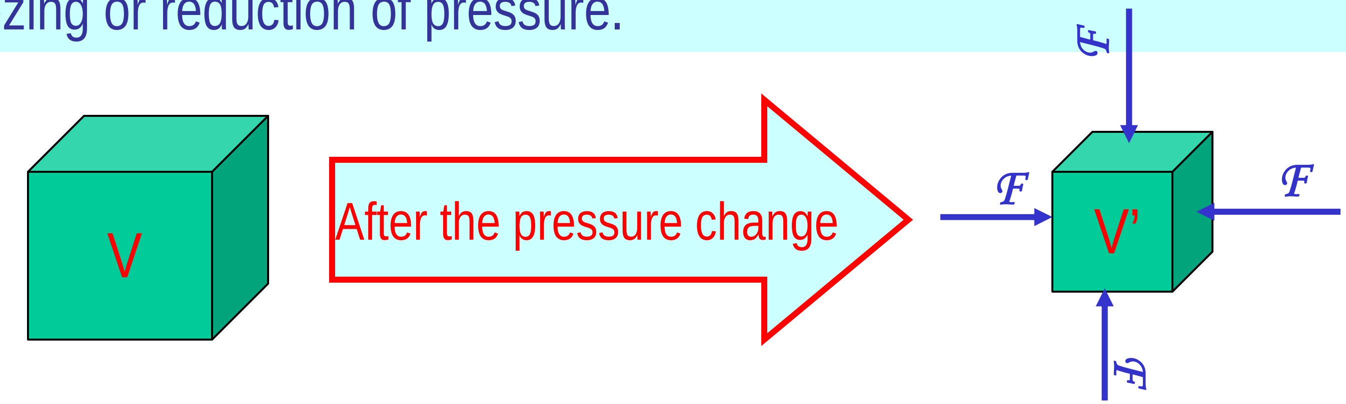
Experimental Observations

1. For fixed external force, the change in length is proportional to the original length
2. The necessary force to produce a given strain is proportional to the cross sectional area

Elastic limit: Maximum stress that can be applied to the substance before it becomes permanently deformed

Bulk Modulus

Bulk Modulus characterizes the response of a substance to uniform squeezing or reduction of pressure.



Volume stress =pressure

$$\text{Pressure} \equiv \frac{\text{Normal Force}}{\text{Surface Area the force applies}} = \frac{F}{A}$$

If the pressure on an object changes by ΔP=ΔF/A, the object will undergo a volume change ΔV.

Bulk Modulus is defined as

$$B \equiv \frac{\text{Volume Stress}}{\text{Volume Strain}} = -\frac{\Delta F / A}{\Delta V / V_i} = -\frac{\Delta P}{\Delta V / V_i}$$

Because the change of volume is reverse to change of pressure.

Compressibility is the reciprocal of Bulk Modulus

Example for Solid's Elastic Property

A solid brass sphere is initially under normal atmospheric pressure of $1.0\text{x}10^5 \text{N/m}^2$. The sphere is lowered into the ocean to a depth at which the pressures is $2.0\text{x}10^7 \text{N/m}^2$. The volume of the sphere in air is 0.5m^3. By how much its volume change once the sphere is submerged?

Since bulk modulus is $$\mathrm{B} = -\frac{\Delta P}{\Delta V / V_i}$$

The amount of volume change is $$\Delta V = -\frac{\Delta P V_i}{\mathrm{B}}$$

From table 12.1, bulk modulus of brass is $6.1\text{x}10^{10}$ N/m^2

The pressure change ΔP is $$\Delta P = P_f - P_i = 2.0\times10^7 - 1.0\times10^5 \approx 2.0\times10^7$$

Therefore the resulting volume change ΔV is $$\Delta V = V_f - V_i = -\frac{2.0\times10^7 \times 0.5}{6.1\times10^{10}} = -1.6\times10^{-4}\, m^3$$

The volume has decreased.

Density and Specific Gravity

Density, ρ (rho) , of an object is defined as mass per unit volume

$$\rho \equiv \frac{M}{V}$$

Unit? kg/m^3

Dimension? $[ML^{-3}]$

Specific Gravity of a substance is defined as the ratio of the density of the substance to that of water at 4.0 °C (ρ_{H2O}=1.00g/cm^3).

$$SG \equiv \frac{\rho_{\text{substance}}}{\rho_{H_2O}}$$

Unit? None

Dimension? None

What do you think would happen of a substance in the water dependent on SG?

$SG > 1$ Sink in the water

$SG < 1$ Float on the surface

Fluid and Pressure

What are the three states of matter?

Solid, Liquid, and Gas

How do you distinguish them?

By the time it takes for a particular substance to change its shape in reaction to external forces.

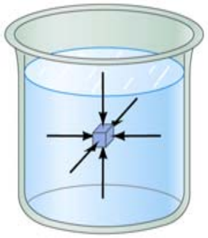
What is a fluid?

A collection of molecules that are randomly arranged and loosely bound by forces between them or by the external container.

We will first learn about mechanics of fluid at rest, *fluid statics*.

In what way do you think fluid exerts stress on the object submerged in it?

Fluid cannot exert shearing or tensile stress. Thus, the only force the fluid exerts on an object immersed in it is the forces perpendicular to the surfaces of the object.



This force by the fluid on an object usually is expressed in the form of the force on a unit area at the given depth, the pressure, defined as

$$P \equiv \frac{F}{A}$$

Expression of pressure for an infinitesimal area dA by the force dF is

$$P = \frac{dF}{dA}$$

Note that pressure is a scalar quantity because it's the magnitude of the force on a surface area A.

What is the unit and dimension of pressure?

Unit: N/m^2

Dim.: $[M][L^{-1}][T^{-2}]$

Special SI unit for pressure is Pascal

$$1Pa \equiv 1N/m^2$$

Example for Pressure

The mattress of a water bed is 2.00m long by 2.00m wide and 30.0cm deep. a) Find the weight of the water in the mattress.

The volume density of water at the normal condition (0^{o}C and 1 atm) is 1000kg/m^3. So the total mass of the water in the mattress is

$$m = \rho_W V_M = 1000 \times 2.00 \times 2.00 \times 0.300 = 1.20 \times 10^3 kg$$

Therefore the weight of the water in the mattress is

$$W = mg = 1.20 \times 10^3 \times 9.8 = 1.18 \times 10^4 N$$

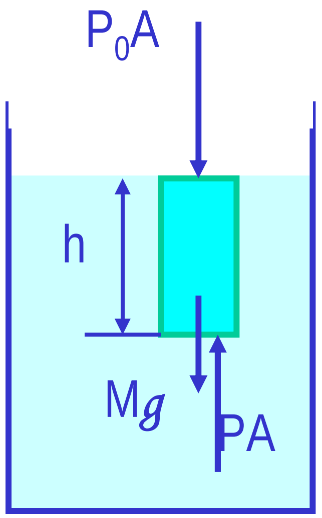
b) Find the pressure exerted by the water on the floor when the bed rests in its normal position, assuming the entire lower surface of the mattress makes contact with the floor.

Since the surface area of the mattress is 4.00 m^2, the pressure exerted on the floor is

$$P = \frac{F}{A} = \frac{mg}{A} = \frac{1.18 \times 10^4}{4.00} = 2.95 \times 10^3$$

Variation of Pressure and Depth

Water pressure increases as a function of depth, and the air pressure decreases as a function of altitude. Why?



It seems that the pressure has a lot to do with the total mass of the fluid above the object that puts weight on the object.

Let's imagine a liquid contained in a cylinder with height h and cross sectional area A immersed in a fluid of density ρ at rest, as shown in the figure, and the system is in its equilibrium.

If the liquid in the cylinder is the same substance as the fluid, the mass of the liquid in the cylinder is $M = \rho V = \rho A h$

Since the system is in its equilibrium $PA - P_0 A - Mg = PA - P_0 A - \rho A h g = 0$

Therefore, we obtain $P = P_0 + \rho g h$

Atmospheric pressure P_0 is

$$1.00\,atm = 1.013 \times 10^5\,Pa$$

The pressure at the depth h below the surface of a fluid open to the atmosphere is greater than atmospheric pressure by $\rho g h$.

What else can you learn from this?

Pascal's Principle and Hydraulics

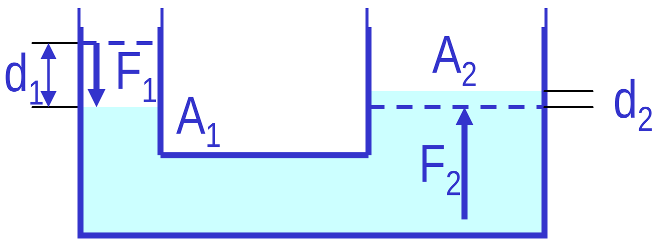
A change in the pressure applied to a fluid is transmitted undiminished to every point of the fluid and to the walls of the container.

$$P = P_0 + \rho g h$$

What happens if P_0 is changed?

The resultant pressure P at any given depth h increases as much as the change in P_0.

This is the principle behind hydraulic pressure. How?



Since the pressure change caused by the the force F_1 applied on to the area A_1 is transmitted to the F_2 on an area A_2.

$$P = \frac{F_1}{A_1} = \frac{F_2}{A_2}$$

Therefore, the resultant force F_2 is

$$F_2 = \frac{A_2}{A_1} F_1$$

In other words, the force gets multiplied by the ratio of the areas A_2/A_1 and is transmitted to the force F_2 on the surface.

This seems to violate some kind of conservation law, doesn't it?

No, the actual displaced volume of the fluid is the same. And the work done by the forces are still the same.

$$F_2 = \frac{d_1}{d_2} F_1$$

Example for Pascal's Principle

In a car lift used in a service station, compressed air exerts a force on a small piston that has a circular cross section and a radius of 5.00cm. This pressure is transmitted by a liquid to a piston that has a radius of 15.0cm. What force must the compressed air exert to lift a car weighing 13,300N? What air pressure produces this force?

Using the Pascal's principle, one can deduce the relationship between the forces, the force exerted by the compressed air is

$$F_1 = \frac{A_2}{A_1} F_2 = \frac{\pi (0.15)^2}{\pi (0.05)^2} \times 1.33 \times 10^4 = 1.48 \times 10^3\, N$$

Therefore the necessary pressure of the compressed air is

$$P = \frac{F_1}{A_1} = \frac{1.48 \times 10^3}{\pi (0.05)^2} = 1.88 \times 10^5\, Pa$$

Example for Pascal's Principle

Estimate the force exerted on your eardrum due to the water above when you are swimming at the bottom of the pool with a depth 5.0 m.

We first need to find out the pressure difference that is being exerted on the eardrum. Then estimate the area of the eardrum to find out the force exerted on the eardrum.

Since the outward pressure in the middle of the eardrum is the same as normal air pressure

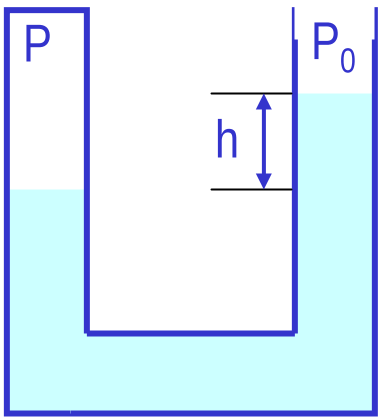
$$P - P_0 = \rho_W gh = 1000 \times 9.8 \times 5.0 = 4.9 \times 10^4 Pa$$

Estimating the surface area of the eardrum at $1.0cm^2 = 1.0x10^{-4}$ m^2, we obtain

$$F = (P - P_0)A \approx 4.9 \times 10^4 \times 1.0 \times 10^{-4} \approx 4.9 N$$

Absolute and Relative Pressure

How can one measure pressure?



One can measure pressure using an open-tube manometer, where one end is connected to the system with unknown pressure P and the other open to air with pressure P_0.

The measured pressure of the system is $P = P_0 + \rho g h$

This is called the absolute pressure, because it is the actual value of the system's pressure.

In many cases we measure pressure difference with respect to atmospheric pressure due to changes in P_0 depending on the environment. This is called gauge or relative pressure.

$$P_G = P - P_0 = \rho g h$$

The common barometer which consists of a mercury column with one end closed at vacuum and the other open to the atmosphere was invented by Evangelista Torricelli.

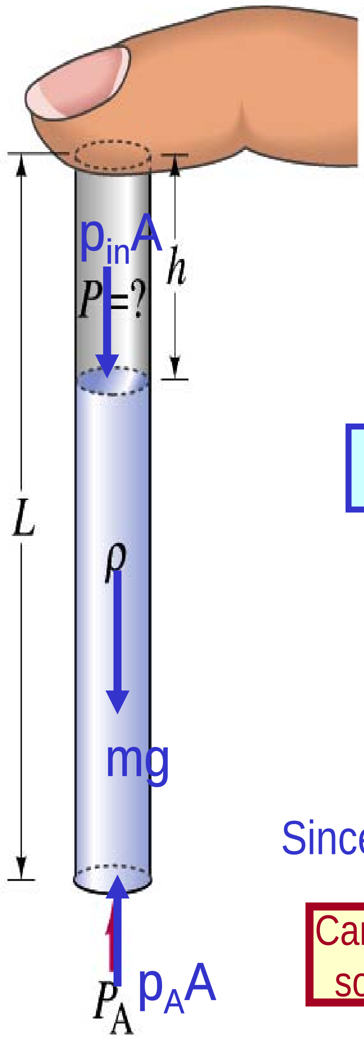
Since the closed end is at vacuum, it does not exert any force. 1 atm is

$$P_0 = \rho g h = (13.595 \times 10^3 \, kg/m^3)(9.80665 \, m/s^2)(0.7600 \, m)$$

$$= 1.013 \times 10^5 \, Pa = 1 \, atm$$

If one measures the tire pressure with a gauge at 220kPa the actual pressure is 101kPa+220kPa=303kPa.

Finger Holds Water in Straw



You insert a straw of length *L* into a tall glass of your favorite beverage. You place your finger over the top of the straw so that no air can get in or out, and then lift the straw from the liquid. You find that the straw strains the liquid such that the distance from the bottom of your finger to the top of the liquid is *h*. Does the air in the space between your finder and the top of the liquid have a pressure P that is (a) greater than, (b) equal to, or (c) less than, the atmospheric pressure P_A outside the straw?

Less

What are the forces in this problem?

Gravitational force on the mass of the liquid $F_g = mg = \rho A(L-h)g$

Force exerted on the top surface of the liquid by inside air pressure $F_{in} = p_{in}A$

Force exerted on the bottom surface of the liquid by outside air $F_{out} = -p_A A$

Since it is at equilibrium $F_{out} + F_g + F_{in} = 0$ ⇒ $-p_A A + \rho g(L-h)A + p_{in}A = 0$

Cancel A and solve for p_{in} ⇒ $p_{in} = p_A - \rho g(L-h)$

So p_{in} is less than P_A by ρgh.

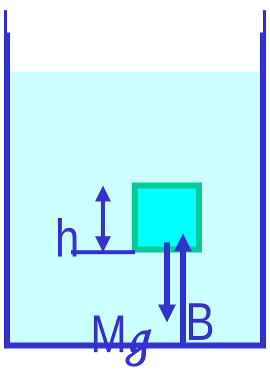
Buoyant Forces and Archimedes' Principle

Why is it so hard to put a beach ball under water while a piece of small steel sinks in the water?

The water exerts force on an object immersed in the water.
This force is called Buoyant force.

How does the Buoyant force work? The magnitude of the buoyant force always equals the weight of the fluid in the volume displaced by the submerged object.

This is called, Archimedes' principle. What does this mean?



Let's consider a cube whose height is h and is filled with fluid and at its equilibrium. Then the weight Mg is balanced by the buoyant force B.

$$B = F_g = Mg$$

And the pressure at the bottom of the cube is larger than the top by ρgh.

Therefore,

$$\Delta P = B / A = \rho gh$$

$$B = \Delta PA = \rho ghA = \rho Vg$$

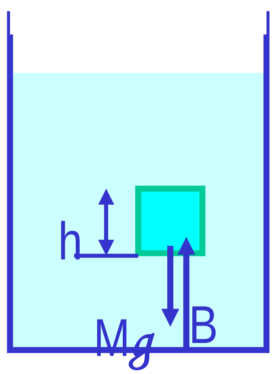
$$B = F_g = \rho Vg = Mg$$

Where Mg is the weight of the fluid.

More Archimedes' Principle

Let's consider buoyant forces in two special cases.

Case 1: Totally submerged object



Let's consider an object of mass M, with density ρ_0, is immersed in the fluid with density ρ_f.

The magnitude of the buoyant force is $B = \rho_f V g$

The weight of the object is $F_g = Mg = \rho_0 V g$

Therefore total force of the system is $F = B - F_g = (\rho_f - \rho_0) V g$

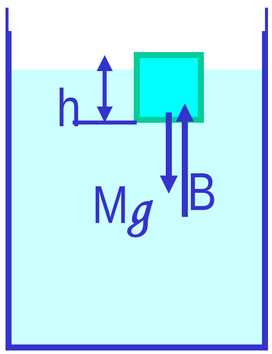
What does this tell you?

The total force applies to different directions, depending on the difference of the density between the object and the fluid.

1. If the density of the object is smaller than the density of the fluid, the buoyant force will push the object up to the surface.
2. If the density of the object is larger that the fluid's, the object will sink to the bottom of the fluid.

More Archimedes' Principle

Case 2: Floating object



Let's consider an object of mass M, with density ρ_0, is in static equilibrium floating on the surface of the fluid with density ρ_f, and the volume submerged in the fluid is V_f.

The magnitude of the buoyant force is $B = \rho_f V_f g$

The weight of the object is $F_g = Mg = \rho_0 V_0 g$

Therefore total force of the system is $F = B - F_g = \rho_f V_f g - \rho_0 V_0 g = 0$

Since the system is in static equilibrium

$$\rho_f V_f g = \rho_0 V_0 g$$

$$\frac{\rho_0}{\rho_f} = \frac{V_f}{V_0}$$

What does this tell you?

Since the object is floating its density is always smaller than that of the fluid.

The ratio of the densities between the fluid and the object determines the submerged volume under the surface.

Example for Archimedes' Principle

Archimedes was asked to determine the purity of the gold used in the crown. The legend says that he solved this problem by weighing the crown in air and in water. Suppose the scale read 7.84N in air and 6.86N in water. What should he have to tell the king about the purity of the gold in the crown?

In the air the tension exerted by the scale on the object is the weight of the crown

$$T_{air} = mg = 7.84N$$

In the water the tension exerted by the scale on the object is

$$T_{water} = mg - B = 6.86N$$

Therefore the buoyant force B is

$$B = T_{air} - T_{water} = 0.98N$$

Since the buoyant force B is

$$B = \rho_w V_w g = \rho_w V_c g = 0.98N$$

The volume of the displaced water by the crown is

$$V_c = V_w = \frac{0.98N}{\rho_w g} = \frac{0.98}{1000 \times 9.8} = 1.0 \times 10^{-4} m^3$$

Therefore the density of the crown is

$$\rho_c = \frac{m_c}{V_c} = \frac{m_c g}{V_c g} = \frac{7.84}{V_c g} = \frac{7.84}{1.0 \times 10^{-4} \times 9.8} = 8.3 \times 10^3 kg/m^3$$

Since the density of pure gold is $19.3 \times 10^3 kg/m^3$, this crown is either not made of pure gold or hollow.

Example for Buoyant Force

What fraction of an iceberg is submerged in the sea water?

Let's assume that the total volume of the iceberg is V_i. Then the weight of the iceberg F_{gi} is

$$F_{gi} = \rho_i V_i g$$

Let's then assume that the volume of the iceberg submerged in the sea water is V_w. The buoyant force B caused by the displaced water becomes

$$B = \rho_w V_w g$$

Since the whole system is at its static equilibrium, we obtain

$$\rho_i V_i g = \rho_w V_w g$$

Therefore the fraction of the volume of the iceberg submerged under the surface of the sea water is

$$\frac{V_w}{V_i} = \frac{\rho_i}{\rho_w} = \frac{917\, kg/m^3}{1030\, kg/m^3} = 0.890$$

About 90% of the entire iceberg is submerged in the water!!!

Flow Rate and the Equation of Continuity

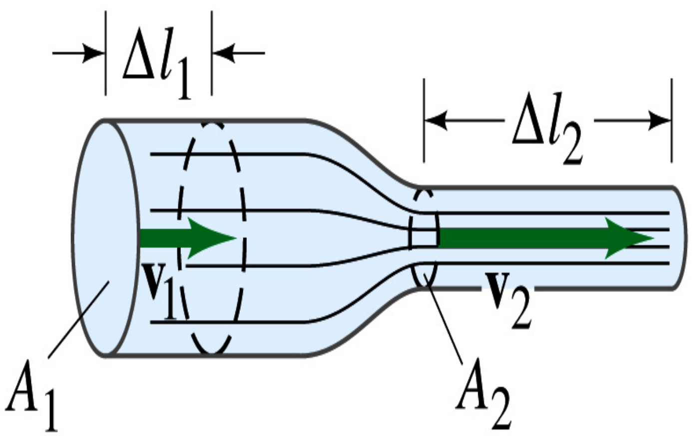
Study of fluid in motion: Fluid Dynamics

If the fluid is water: Water dynamics?? Hydro-dynamics

Two main types of flow

- **Streamline or Laminar flow**: Each particle of the fluid follows a smooth path, a streamline w/o friction
- **Turbulent flow**: Erratic, small, whirlpool-like circles called eddy current or eddies which absorbs a lot of energy

Flow rate: the mass of fluid that passes a given point per unit time $\Delta m / \Delta t$



$$\frac{\Delta m_1}{\Delta t} = \frac{\rho_1 \Delta V_1}{\Delta t} = \frac{\rho_1 A_1 \Delta l_1}{\Delta t} = \rho_1 A_1 v_1$$

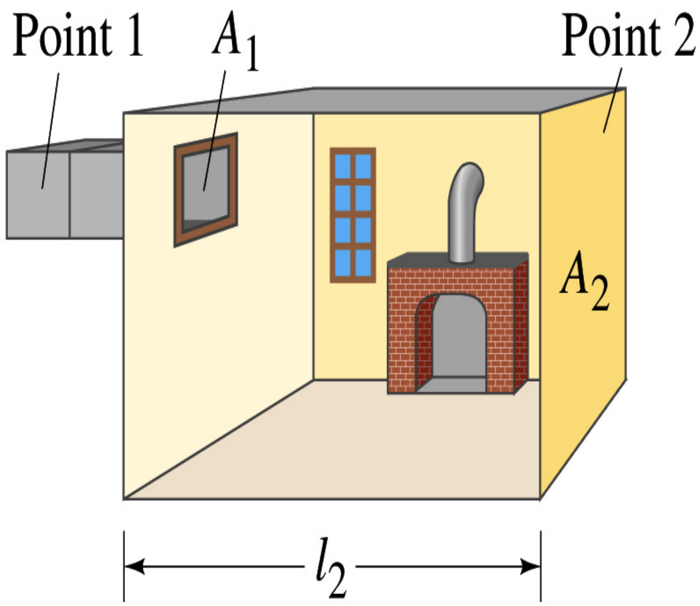
since the total flow must be conserved

$$\frac{\Delta m_1}{\Delta t} = \frac{\Delta m_2}{\Delta t} \Rightarrow \rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

Equation of Continuity

Example for Equation of Continuity

How large must a heating duct be if air moving at 3.0m/s along it can replenish the air every 15 minutes, in a room of 300m^3 volume? Assume the air's density remains constant.



Using equation of continuity

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

Since the air density is constant

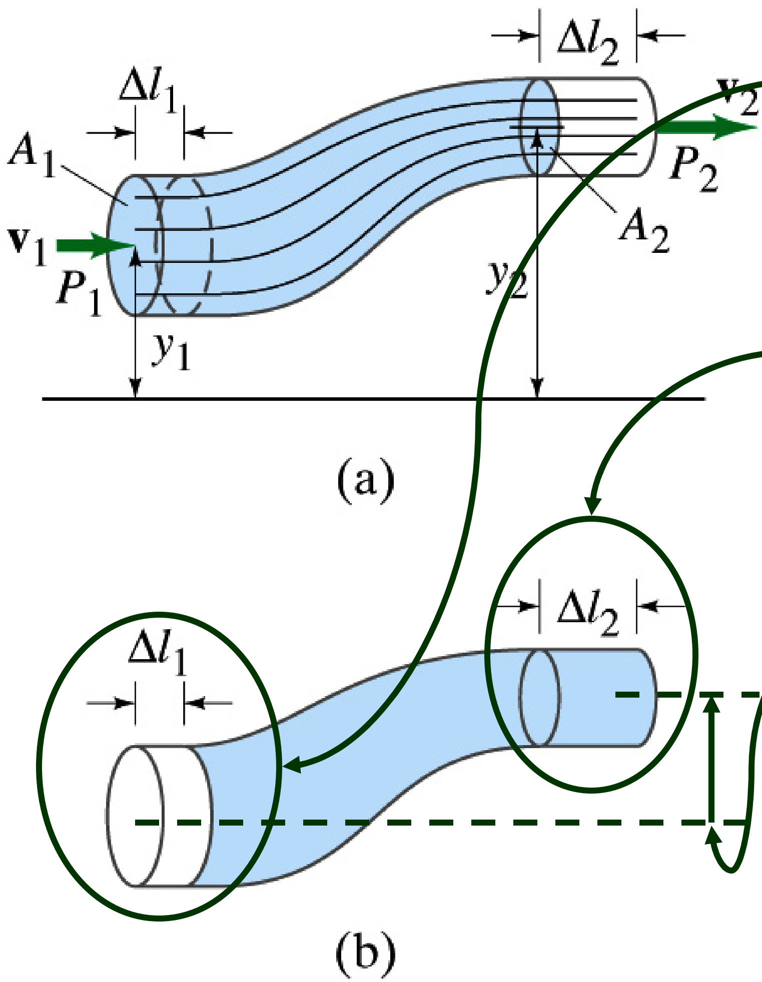
$$A_1 v_1 = A_2 v_2$$

Now let's imagine the room as the large section of the duct

$$A_1 = \frac{A_2 v_2}{v_1} = \frac{A_2 l_2 / t}{v_1} = \frac{V_2}{v_1 \cdot t} = \frac{300}{3.0 \times 900} = 0.11 m^2$$

Bernoulli's Equation

Bernoulli's Principle: Where the velocity of fluid is high, the pressure is low, and where the velocity is low, the pressure is high.



Amount of work done by the force, F_1, that exerts pressure, P_1, at point 1

$$W_1 = F_1 \Delta l_1 = P_1 A_1 \Delta l_1$$

Amount of work done on the other section of the fluid is

$$W_2 = -P_2 A_2 \Delta l_2$$

Work done by the gravitational force to move the fluid mass, m, from y_1 to y_2 is

$$W_3 = -mg\left(y_2 - y_1\right)$$

Bernoulli's Equation cont'd

Since

$$\frac{1}{2}\rho A_2\Delta l_2 v_2^2 - \frac{1}{2}\rho A_1\Delta l_1 v_1^2 = P_1 A_1\Delta l_1 - P_2 A_2\Delta l_2 - \rho A_2\Delta l_2 g y_2 + \rho A_1\Delta l_1 g y_1$$

We obtain

$$\frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2 = P_1 - P_2 - \rho g y_2 + \rho g y_1$$

Re-organize

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

Bernoulli's Equation

Thus, for any two points in the flow

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = \ const.$$

For static fluid

$$P_2 = P_1 + \rho g\left(y_1 - y_2\right) = P_1 + \rho g h$$

Pascal's Law

For the same heights

$$P_2 = P_1 + \frac{1}{2}\rho\left(v_1^2 - v_2^2\right)$$

The pressure at the faster section of the fluid is smaller than slower section.

Example for Bernoulli's Equation

Water circulates throughout a house in a hot-water heating system. If the water is pumped at a speed of 0.5m/s through a 4.0cm diameter pipe in the basement under a pressure of 3.0atm, what will be the flow speed and pressure in a 2.6cm diameter pipe on the second 5.0m above? Assume the pipes do not divide into branches.

Using the equation of continuity, flow speed on the second floor is

$$v_2 = \frac{A_1 v_1}{A_2} = \frac{\pi r_1^2 v_1}{\pi r_2^2} = 0.5 \times \left(\frac{0.020}{0.013}\right)^2 = 1.2 m/s$$

Using Bernoulli's equation, the pressure in the pipe on the second floor is

$$P_2 = P_1 + \frac{1}{2}\rho\left(v_1^2 - v_2^2\right) + \rho g\left(y_1 - y_2\right)$$

$$= 3.0\times10^5 + \frac{1}{2}1\times10^3\left(0.5^2 - 1.2^2\right) + 1\times10^3\times9.8\times(-5)$$

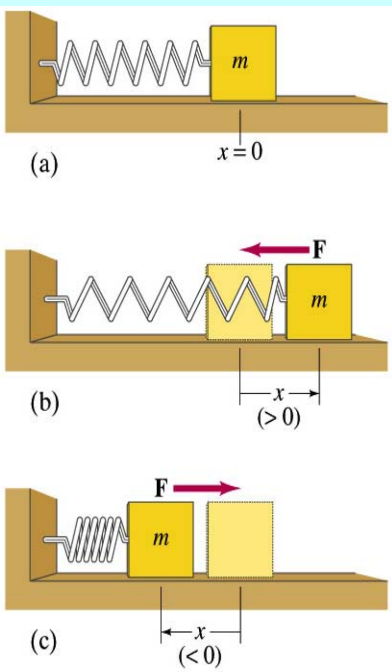
$$= 2.5\times10^5 N/m^2$$

Vibration or Oscillation

What are the things that vibrate/oscillate?

- Tuning fork
- A pendulum
- A car going over a bump
- Building and bridges
- The spider web with a prey

So what is a vibration or oscillation? A periodic motion that repeats over the same path.



A simplest case is a block attached at the end of a coil spring.

When a spring is stretched from its equilibrium position by a length x, the force acting on the mass is

$$F = -kx$$

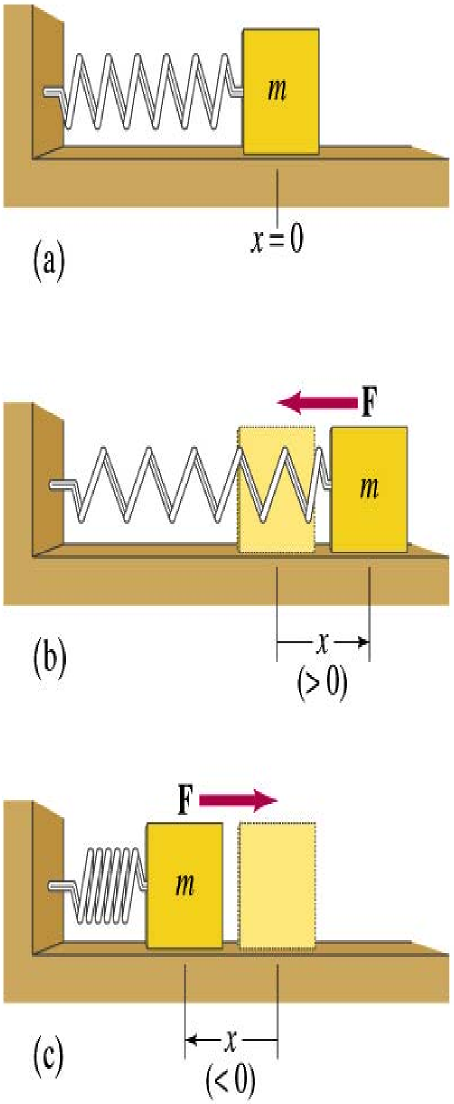
The sign is negative, because the force resists against the change of length, directed toward the equilibrium position.

Acceleration is proportional to displacement from the equilibrium

Acceleration is opposite direction to displacement

This system is doing a simple harmonic motion (SHM).

Vibration or Oscillation Properties



The maximum displacement from the equilibrium is — Amplitude

One cycle of the oscillation

The complete to-and-fro motion from an initial point

Period of the motion, T

The time it takes to complete one full cycle

Unit? s

Frequency of the motion, f

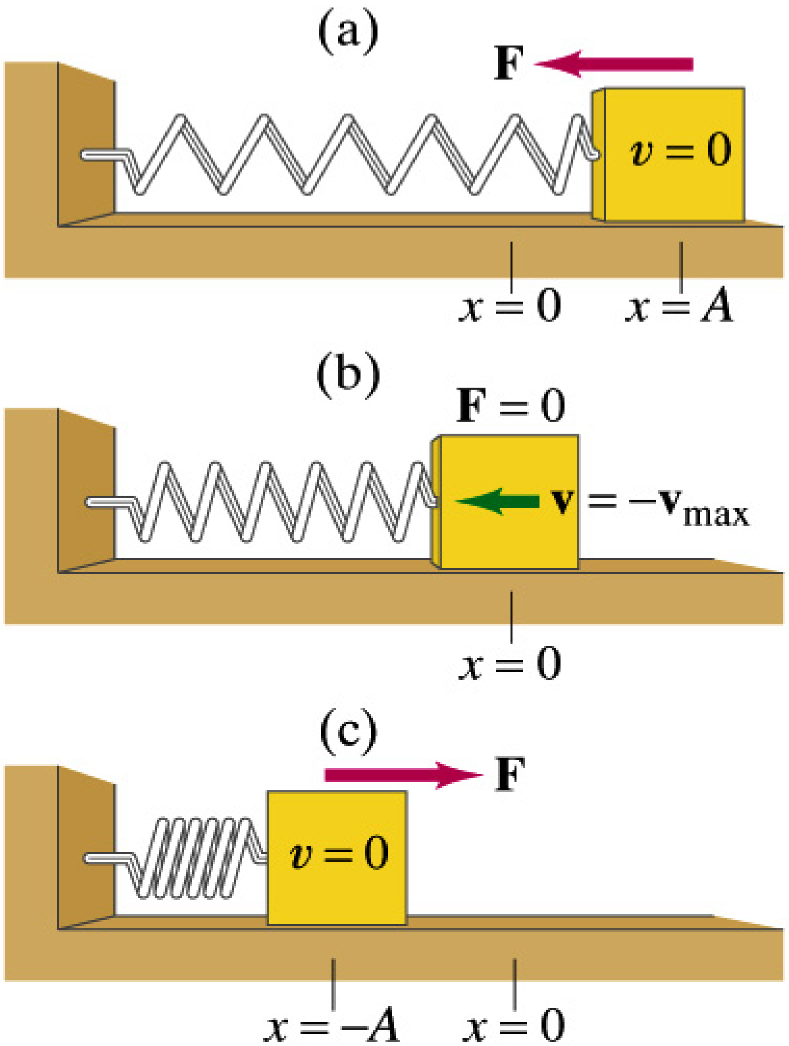
The number of complete cycles per second

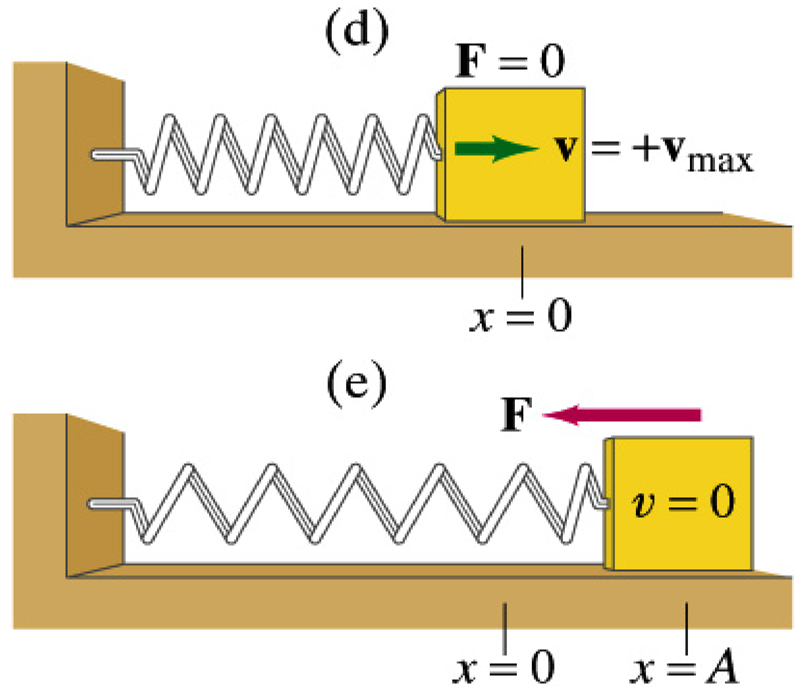
Unit? s^{-1}

Relationship between period and frequency?

$$f = \frac{1}{T} \quad \text{or} \quad T = \frac{1}{f}$$

Vibration or Oscillation Properties





Amplitude? A

- When is the force greatest?
- When is the velocity greatest?
- When is the acceleration greatest?
- When is the potential energy greatest?
- When is the kinetic energy greatest?

Example 11-1

Car springs. When a family of four people with a total mass of 200kg step into their 1200kg car, the car's springs compress 3.0cm. (a) What is the spring constant of the car's spring, assuming they act as a single spring? (b) How far will the car lower if loaded with 300kg?

(a) What is the force on the spring? $F = mg = 200 \cdot 9.8 = 1960N$

From Hooke's law $F = -kx = -k \cdot 0.03 = -mg = 1960N$

$$k = \frac{F_g}{x} = \frac{mg}{x} = \frac{1960}{0.03} = 6.5 \times 10^4 N/m$$

(b) Now that we know the spring constant, we can solve for x in the force equation

$$F = -kx = -mg = -300 \cdot 9.8$$

$$x = \frac{mg}{k} = \frac{300 \cdot 9.8}{6.5 \times 10^4} = 4.5 \times 10^{-2} m$$

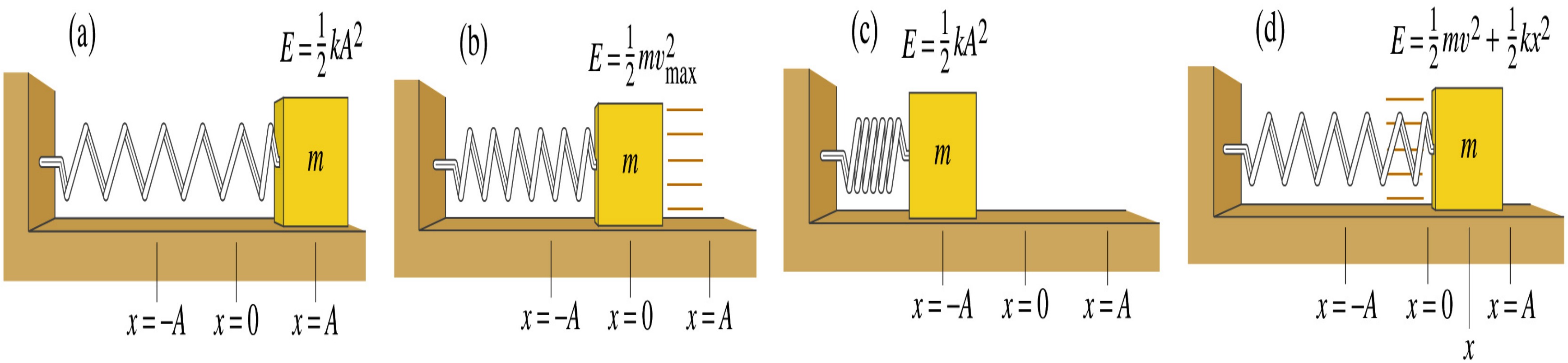
Energy of the Simple Harmonic Oscillator

How do you think the mechanical energy of the harmonic oscillator look without friction?

Kinetic energy of a harmonic oscillator is $KE = \frac{1}{2}mv^2$

The elastic potential energy stored in the spring $PE = \frac{1}{2}kx^2$

Therefore the total mechanical energy of the harmonic oscillator is $E = KE + PE = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$



Total mechanical energy of a simple harmonic oscillator is proportional to the square of the amplitude.

Energy of the Simple Harmonic Oscillator cont'd

Maximum KE is when PE=0

$$KE_{\max} = \frac{1}{2}mv_{\max}^2 = \frac{1}{2}kA^2$$

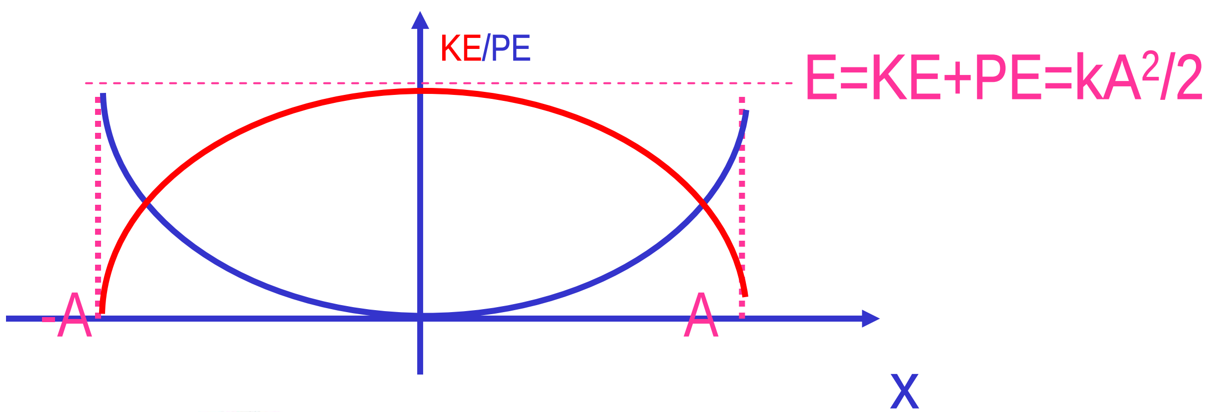
Maximum speed

$$v_{\max} = \sqrt{\frac{k}{m}}A$$

The speed at any given point of the oscillation

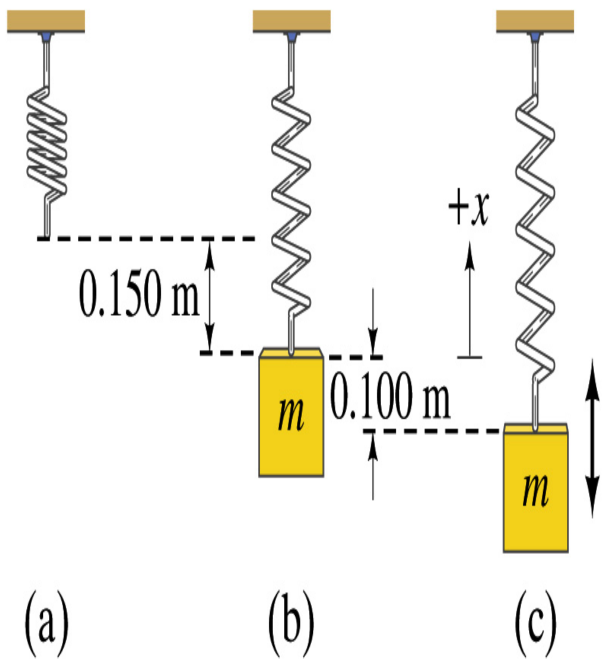
$$E = KE + PE = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

$$v = +\sqrt{k/m\left(A^2 - x^2\right)} = \pm v_{\max}\sqrt{1 - \left(\frac{x}{A}\right)^2}$$



Example 11-3

Spring calculations. A spring stretches 0.150m when a 0.300-kg mass is hung from it. The spring is then stretched an additional 0.100m from this equilibrium position and released.



(a) Determine the spring constant.

From Hooke's law

$$F = -kx = -mg = -0.300 \cdot 9.8N$$

$$k = \frac{mg}{x} = \frac{0.300 \cdot 9.8}{0.150} = 19.6N/m$$

(b) Determine the amplitude of the oscillation.

Since the spring was stretched 0.100m from equilibrium, and is given no initial speed, the amplitude is the same as the additional stretch.

$$A = 0.100m$$

Example cont'd

(c) Determine the maximum velocity v_{max}.

$$v_{max} = \sqrt{\frac{k}{m}}A = \sqrt{\frac{19.6}{0.300}}0.100 = 0.808m/s$$

(d) Determine the magnitude of velocity, v, when the mass is 0.050m from equilibrium.

$$v = \left| v_{max}\sqrt{1-\left(\frac{x}{A}\right)^2} \right| = \left| 0.808\sqrt{1-\left(\frac{0.050}{0.100}\right)^2} \right| = 0.700m/s$$

(d) Determine the magnitude of the maximum acceleration of the mass.

Maximum acceleration is at the point where the mass is stopped to return.

$$F = ma = kA$$

Solve for a

$$a = \frac{kA}{m} = \frac{19.6 \cdot 0.100}{0.300} = 6.53m/s^2$$

Example for Energy of Simple Harmonic Oscillator

A 0.500kg cube connected to a light spring for which the force constant is 20.0 N/m oscillates on a horizontal, frictionless track. a) Calculate the total energy of the system and the maximum speed of the cube if the amplitude of the motion is 3.00 cm.

From the problem statement, A and k are

$$k = 20.0 N/m$$
$$A = 3.00cm = 0.03m$$

The total energy of the cube is

$$E = KE + PE = \frac{1}{2}kA^2 = \frac{1}{2}(20.0)\times(0.03)^2 = 9.00\times 10^{-3} J$$

Maximum speed occurs when kinetic energy is the same as the total energy

$$KE_{\max} = \frac{1}{2}mv_{\max}^2 = E = \frac{1}{2}kA^2$$

$$v_{\max} = A\sqrt{\frac{k}{m}} = 0.03\sqrt{\frac{20.0}{0.500}} = 0.190 m/s$$

Example for Energy of Simple Harmonic Oscillator

b) What is the velocity of the cube when the displacement is 2.00 cm.

velocity at any given displacement is

$$v = \sqrt{k/m\left(A^2 - x^2\right)}$$

$$= \sqrt{20.0 \cdot \left(0.03^2 - 0.02^2\right)/0.500} = 0.141 m/s$$

c) Compute the kinetic and potential energies of the system when the displacement is 2.00 cm.

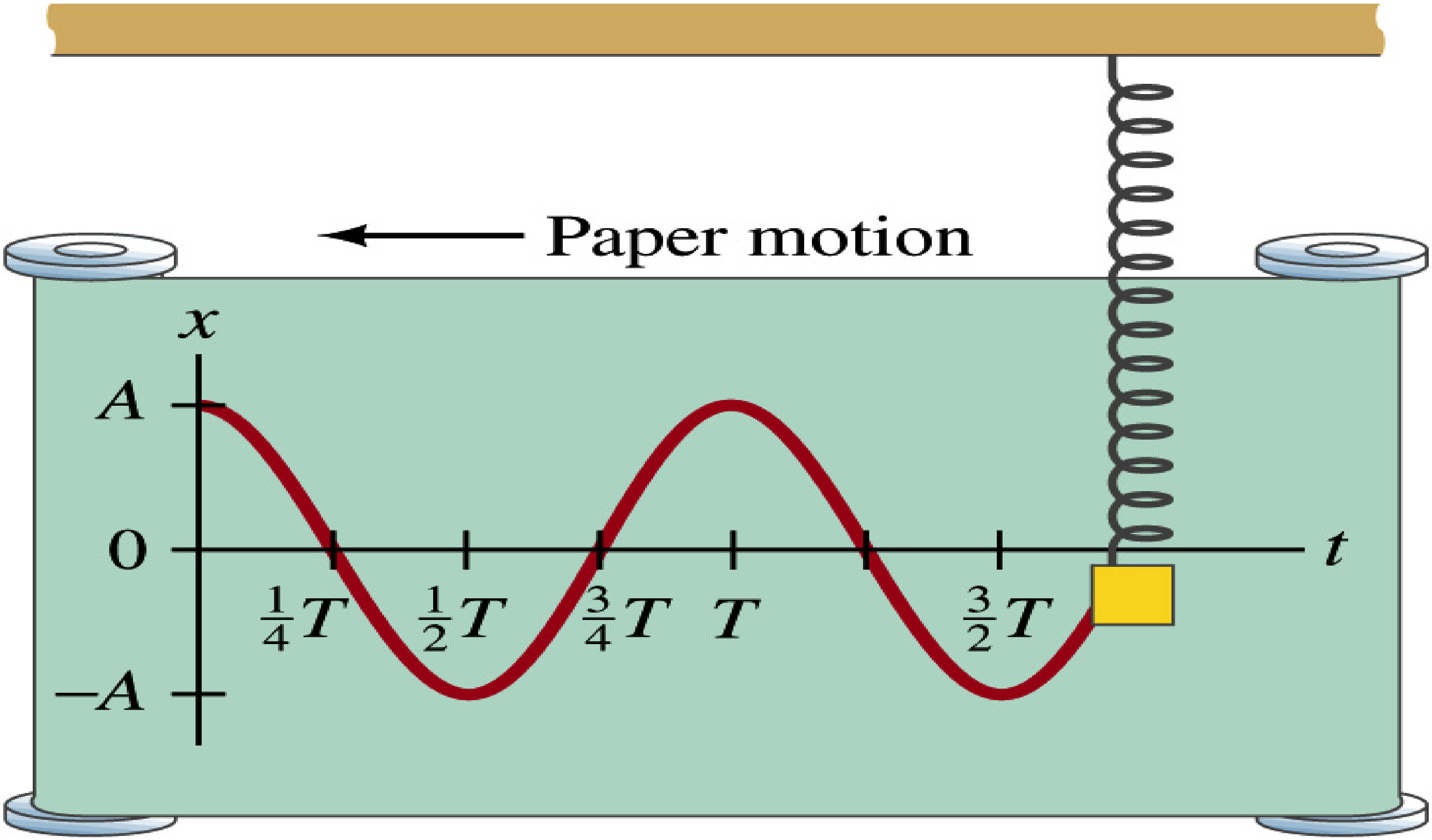
Kinetic energy, KE

$$KE = \frac{1}{2}mv^2 \quad = \frac{1}{2}0.500 \times (0.141)^2 = 4.97 \times 10^{-3} J$$

Potential energy, PE

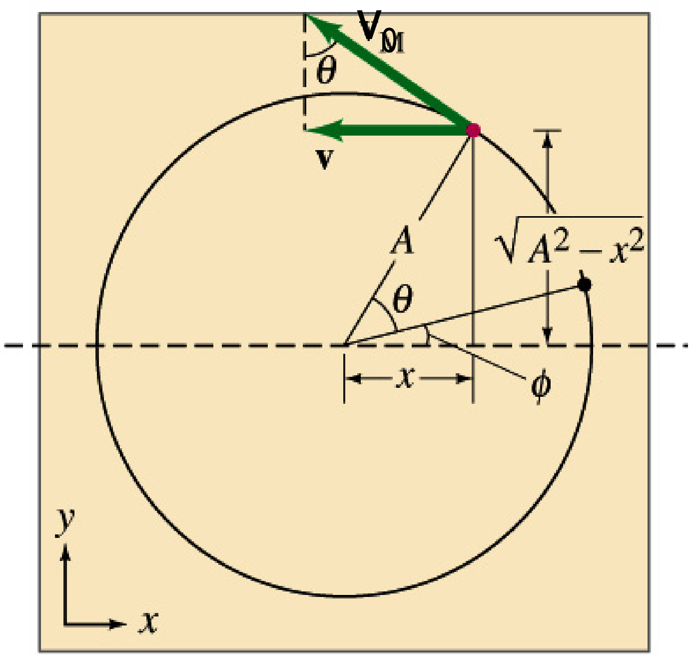
$$PE = \frac{1}{2}kx^2 \quad = \frac{1}{2}20.0 \times (0.02)^2 = 4.00 \times 10^{-3} J$$

Sinusoidal Behavior of SHM



The Period and Sinusoidal Nature of SHM

Consider an object moving on a circle with a constant angular speed ω



$$\sin\theta = \frac{v}{v_0} = \frac{\sqrt{A^2 - x^2}}{A} = \sqrt{1 - \left(\frac{x}{A}\right)^2} \Rightarrow v = v_0\sqrt{1 - \left(\frac{x}{A}\right)^2}$$

Since it takes T to complete one full circular motion

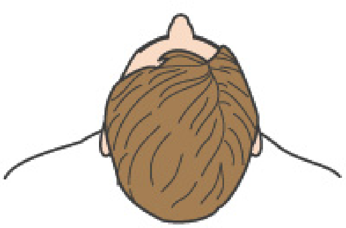
$$v_0 = \frac{2\pi A}{T} = 2\pi A f \Rightarrow T = \frac{2\pi A}{v_0}$$

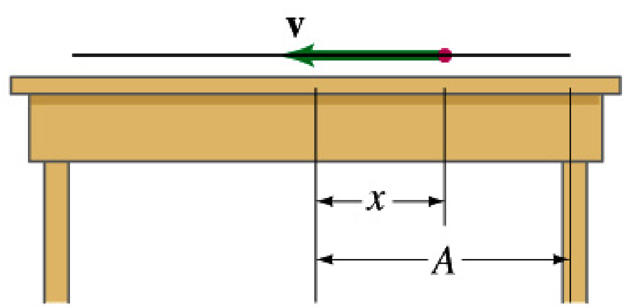
From an energy relationship in a spring SHM

$$\frac{1}{2} m v_0^2 = \frac{1}{2} k A^2 \Rightarrow v_0 = \sqrt{\frac{k}{m}} A$$

Thus, T is

$$T = 2\pi\sqrt{\frac{m}{k}} \Rightarrow f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$





Natural Frequency

If you look at it from the side, it looks as though it is doing a SHM

Example 11-5

Car springs. When a family of four people with a total mass of 200kg step into their 1200kg car, the car's springs compress 3.0cm. The spring constant of the spring is 6.5×10^4N/m. What is the frequency of the car after hitting the bump? Assume that the shock absorber is poor, so the car really oscillates up and down.

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{1400}{6.5\times10^4}} = 0.92s$$

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{6.5\times10^4}{1400}} = 1.09Hz$$

Example 11-6

Spider Web. A small insect of mass 0.30 g is caught in a spider web of negligible mass. The web vibrates predominantly with a frequency of 15Hz. (a) Estimate the value of the spring constant k for the web.

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 15Hz$$

Solve for k

$$k = 4\pi^2 mf^2 = 4\pi^2 \cdot 3\times10^{-4} \cdot (15)^2 = 2.7N/m$$

(b) At what frequency would you expect the web to vibrate if an insect of mass 0.10g were trapped?

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{2.7}{1\times10^{-4}}} = 26Hz$$

The SHM Equation of Motion

The object is moving on a circle with a constant angular speed ω

How is x, its position at any given time expressed with the known quantities?

$$x = A\cos\theta \quad \text{since} \quad \theta = \varpi t \quad \text{and} \quad \varpi = 2\pi f$$

$$x = A\cos\varpi t = A\cos 2\pi ft$$

How about its velocity v at any given time?

$$v_0 = \sqrt{\frac{k}{m}}A$$

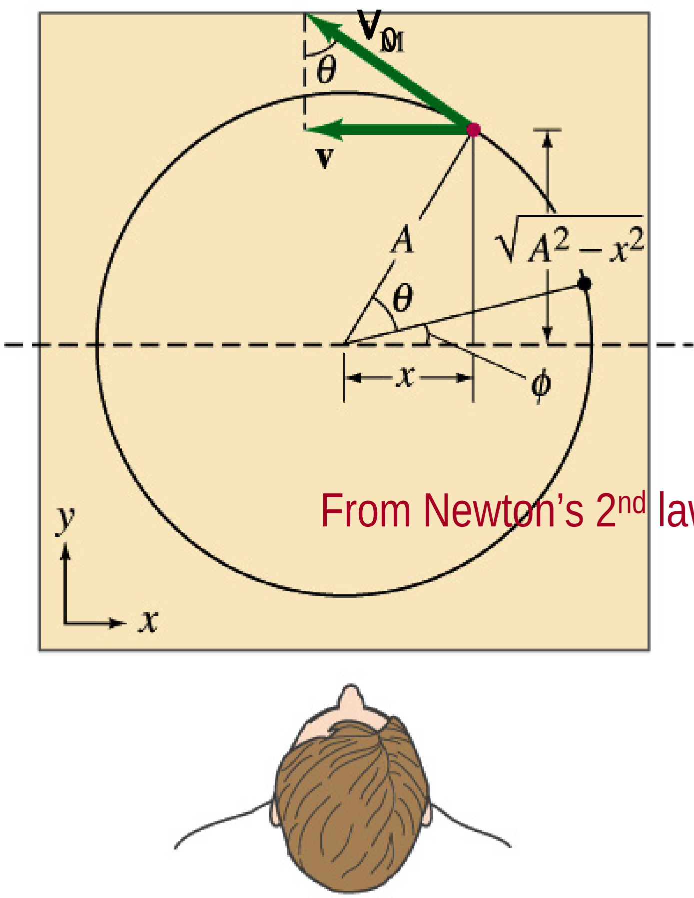
$$v = -v_0\sin\theta = -v_0\sin(\varpi t) = -v_0\sin(2\pi ft)$$

How about its acceleration a at any given time?

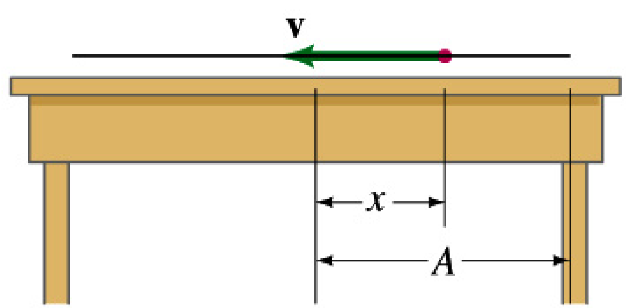
From Newton's 2nd law

$$a = \frac{F}{m} = \frac{-kx}{m} = -\left(\frac{kA}{m}\right)\cos(2\pi ft) = -a_0\cos(2\pi ft)$$

$$a_0 = \frac{kA}{m}$$

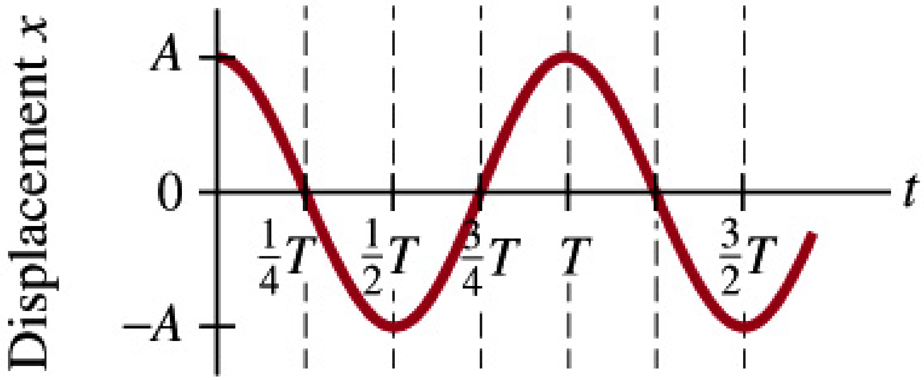


(a)

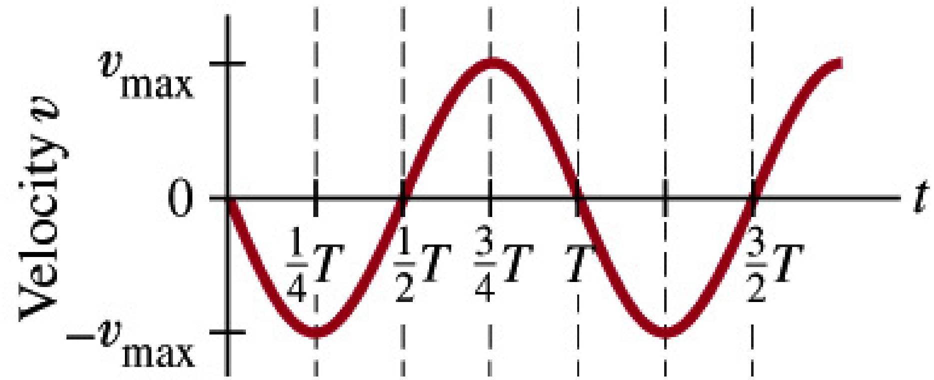


(b)

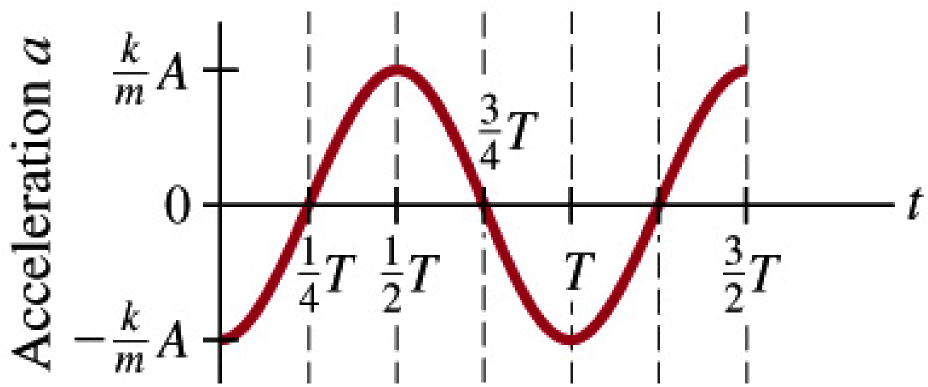
Sinusoidal Behavior of SHM



$$x = A\cos\left(2\pi ft\right)$$

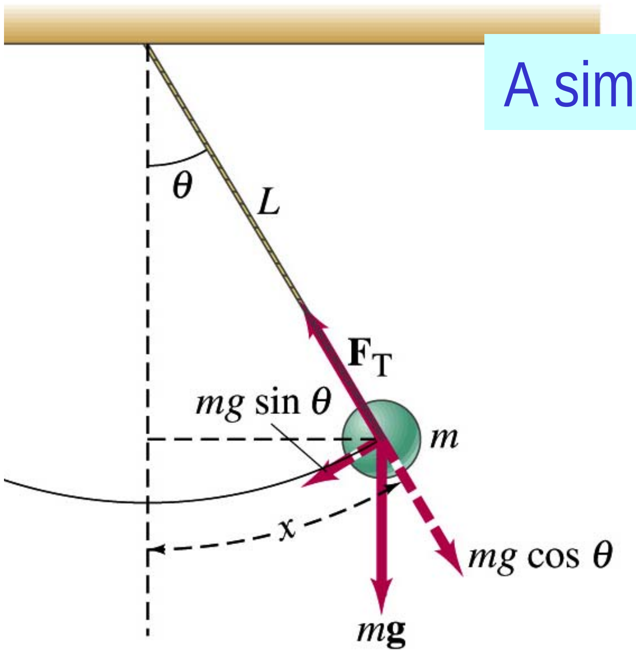


$$v = -v_0 \sin\left(2\pi ft\right)$$



$$a = -a_0 \cos\left(2\pi ft\right)$$

The Simple Pendulum



A simple pendulum also performs periodic motion.

The net force exerted on the bob is

$$\sum F_r = T - mg\cos\theta_A = 0$$

$$\sum F_t = -mg\sin\theta_A \approx -mg\theta$$

Since the arc length, x, is $x = L\theta$

$$F = \sum F_t = -\frac{mg}{L}x$$

Satisfies conditions for simple harmonic motion!

It's almost like Hooke's law with. $k = \frac{mg}{L}$

The period for this motion is $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{mL}{mg}} = 2\pi\sqrt{\frac{L}{g}}$

The period only depends on the length of the string and the gravitational acceleration

Example 11-8

Grandfather clock. (a) Estimate the length of the pendulum in a grandfather clock that ticks once per second.

Since the period of a simple pendulum motion is

$$T = 2\pi\sqrt{\frac{L}{g}}$$

The length of the pendulum in terms of T is

$$L = \frac{T^2 g}{4\pi^2}$$

Thus the length of the pendulum when T=1s is

$$L = \frac{T^2 g}{4\pi^2} = \frac{1\times 9.8}{4\pi^2} = 0.25m$$

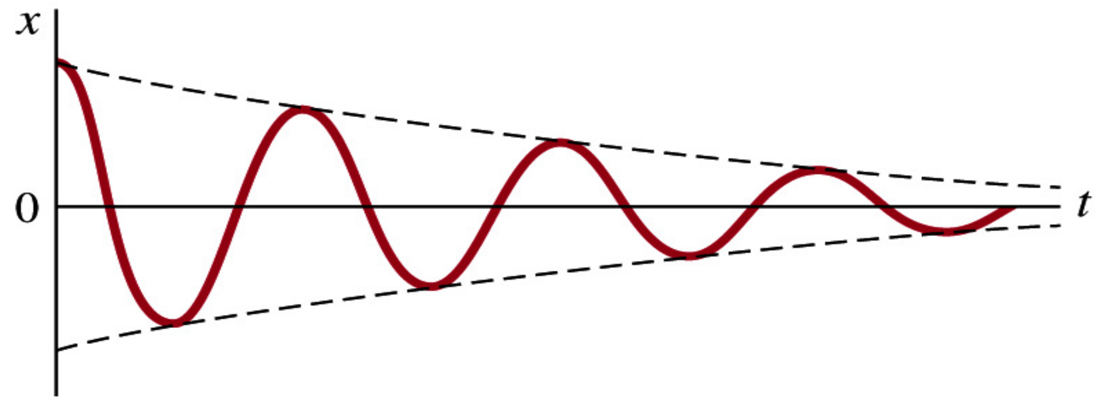
(b) What would be the period of the clock with a 1m long pendulum?

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{1.0}{9.8}} = 2.0s$$

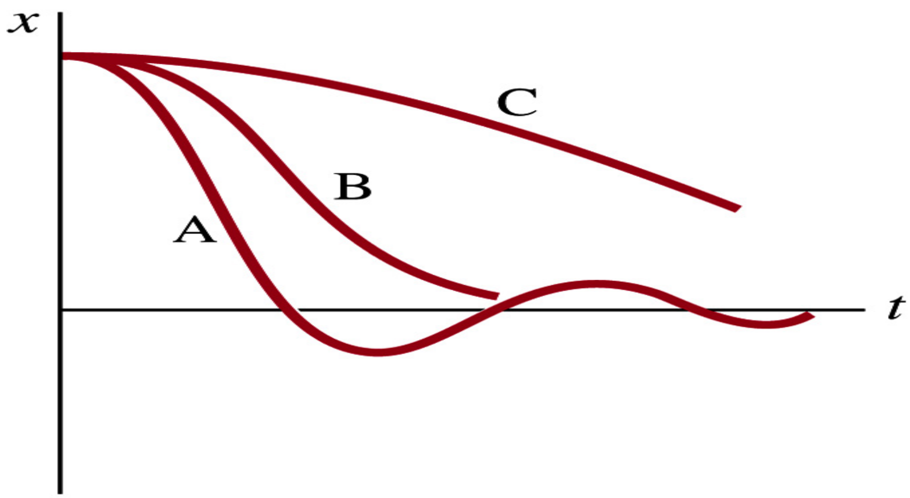
Damped Oscillation

More realistic oscillation where an oscillating object loses its mechanical energy in time by a retarding force such as friction or air resistance.

How do you think the motion would look?



Amplitude gets smaller as time goes on since its energy is spent.



Types of damping

A: Overdamped

B: Critically damped

C: Underdamped

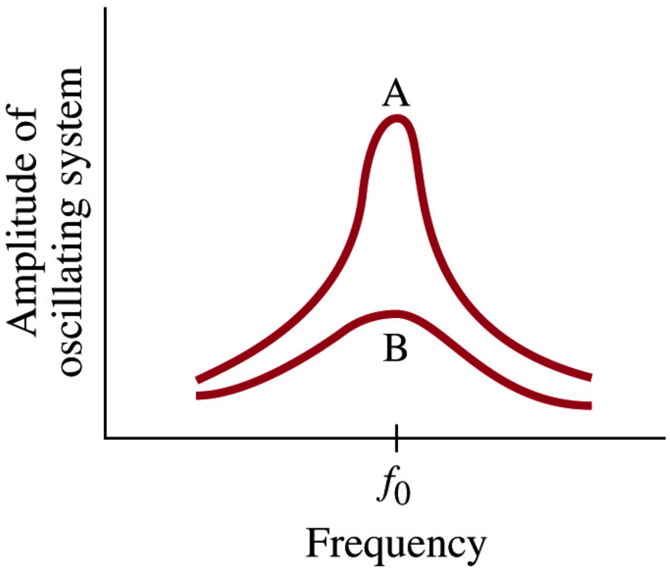
Forced Oscillation; Resonance

When a vibrating system is set into motion, it oscillates with its natural frequency f_0.

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

However a system may have an external force applied to it that has its own particular frequency (f), causing forced vibration.

For a forced vibration, the amplitude of vibration is found to be dependent on the different between f and f_0. and is maximum when $f=f_0$.



A: light damping

B: Heavy damping

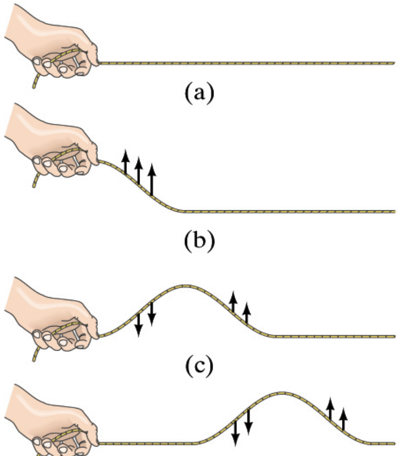
The amplitude can be large when $f=f_0$, as long as damping is small.

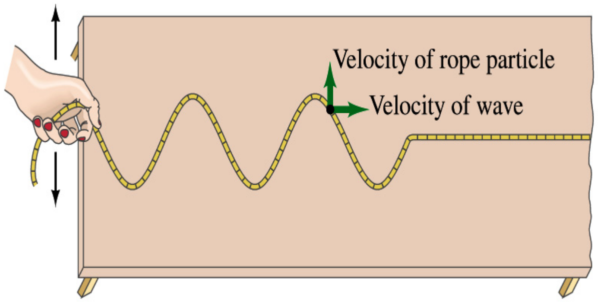
This is called resonance. The natural frequency f_0 is also called resonant frequency.

Wave Motions

- Waves do not move medium rather carry energy from one place to another

- Two forms of waves
  - Pulse
  - Continuous or periodic wave



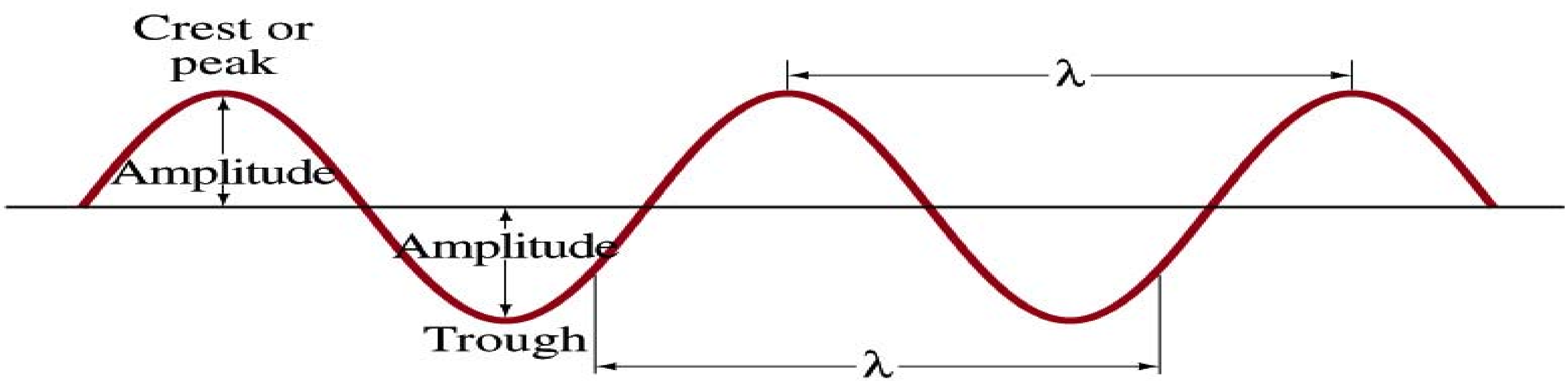


Mechanical Waves

Characterization of Waves

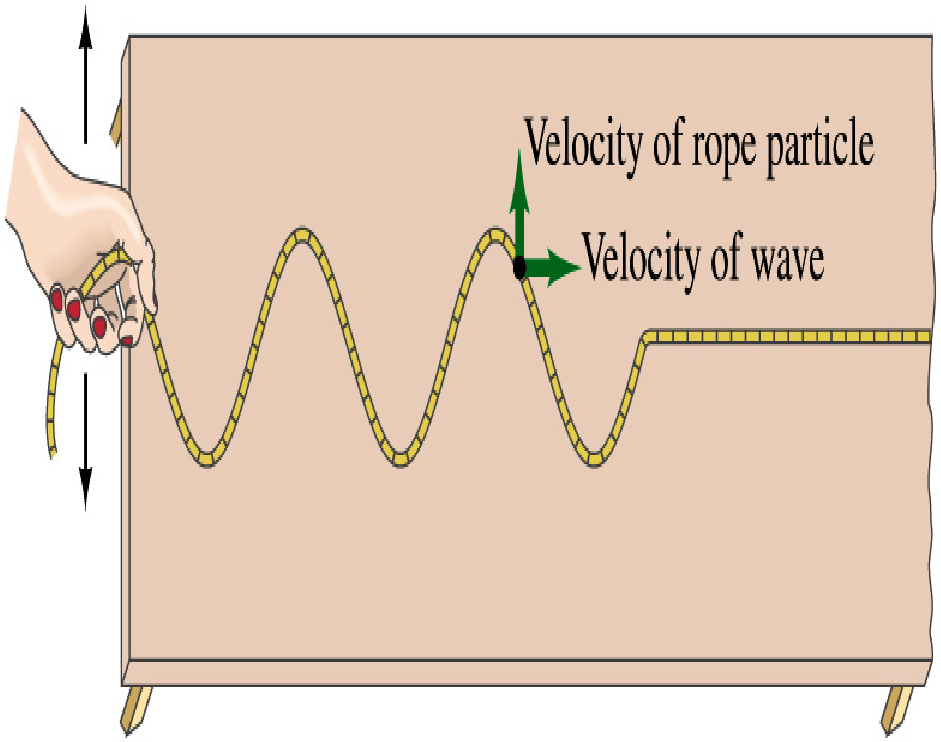
- Waves can be characterized by
  - Amplitude: Maximum height of a crest or the depth of a trough
  - Wave length: Distance between two successive crests or any two identical points on the wave
  - Period: The time elapsed by two successive crests passing by the same point in space.
  - Frequency: Number of crests that pass the same point in space in a unit time
- Wave velocity: The velocity at which any part of the wave moves

$$v = \frac{\lambda}{T} = \lambda f$$



Waves vs Particle Velocity

Is the velocity of a wave moving along a cord the same as the velocity of a particle of the cord?



No. The two velocities are different both in magnitude and direction. The wave on the rope moves to the right but each piece of the rope only vibrates up and down.

Speed of Transverse Waves on Strings

How do we determine the speed of a transverse pulse traveling on a string?

If a string under tension is pulled sideways and released, the tension is responsible for accelerating a particular segment of the string back to the equilibrium position.

So what happens when the tension increases?

The acceleration of the particular segment increases

Which means?

The speed of the wave increases.

Now what happens when the mass per unit length of the string increases?

For the given tension, acceleration decreases, so the wave speed decreases.

Which law does this hypothesis based on?

Newton's second law of motion

Based on the hypothesis we have laid out above, we can construct a hypothetical formula for the speed of wave

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{m/L}}$$

T: Tension on the string

μ: Unit mass per length

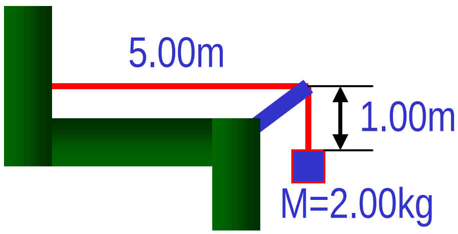
Is the above expression dimensionally sound?

T=kg m/s^2. μ=kg/m

$(T/\mu)^{1/2}=[m^2/s^2]^{1/2}=m/s$

Example for Traveling Wave

A uniform cord has a mass of 0.300kg and a length of 6.00m. The cord passes over a pulley and supports a 2.00kg object. Find the speed of a pulse traveling along this cord.



Since the speed of wave on a string with line density μ and under the tension T is $v = \sqrt{\frac{T}{\mu}}$

The line density μ is $\mu = \frac{0.300\,kg}{6.00\,m} = 5.00 \times 10^{-2} kg/m$

The tension on the string is provided by the weight of the object. Therefore

$$T = Mg = 2.00 \times 9.80 = 19.6 kg \cdot m/s^2$$

Thus the speed of the wave is

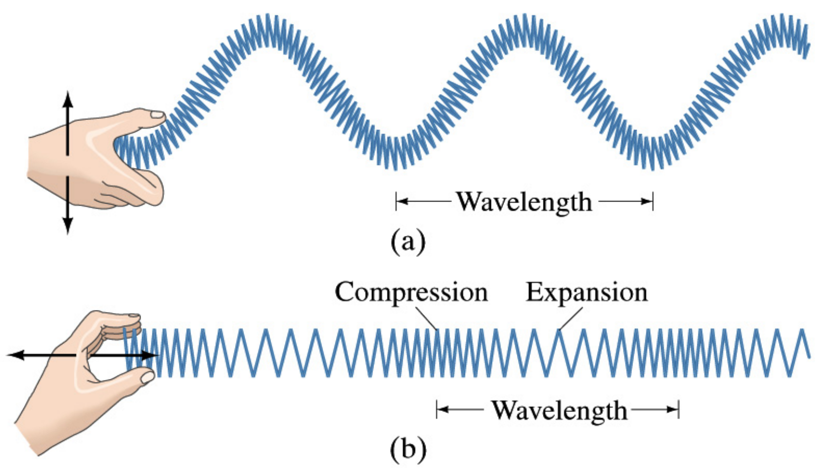
$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{19.6}{5.00 \times 10^{-2}}} = 19.8 m/s$$

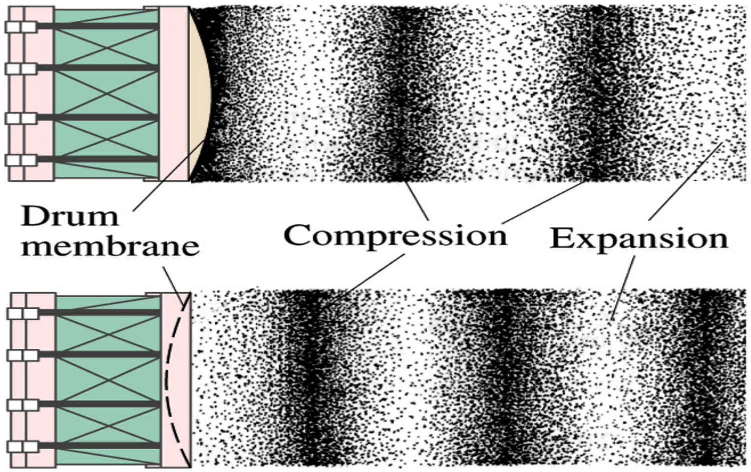
Type of Waves

- Two types of waves
  - Transverse Wave : A wave whose media particles move perpendicular to the direction of the wave
  - Longitudinal wave: A wave whose media particles move along the direction of the wave
    - Speed of a longitudinal wave $v = \sqrt{\frac{\text{Elastic Force Factor}}{\text{inertia factor}}}$

For solid $v = \sqrt{\frac{E}{\rho}}$ E:Young's modulus, ρ: density of solid

liquid/gas $v = \sqrt{\frac{B}{\rho}}$ E: Bulk Modulus, ρ: density





Example 11 – 11

Sound velocity in a steel rail. You can often hear a distant train approaching by putting your ear to the track. How long does it take for the wave to travel down the steel track if the train is 1.0km away?

The speed of the wave is

$$v = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{2.0\times10^{11} N/m^2}{7.8\times10^{3} kg/m^3}} = 5.1\times10^{3} m/s$$

The time it takes for the wave to travel is

$$t = \frac{l}{v} = \frac{1.0\times10^{3} m}{5.1\times10^{3} m/s} = 0.20s$$

Energy Transported by Waves

Waves transport energy from one place to another.

As waves travel through a medium, the energy is transferred as vibrational energy from particle to particle of the medium.

For a sinusoidal wave of frequency f, the particles move in SHM as a wave passes. Thus each particle has an energy

$$E = \frac{1}{2}kA^2$$

Energy transported by a wave is proportional to the square of the amplitude.

Intensity of wave is defined as the power transported across unit area perpendicular to the direction of energy flow.

$$I = \frac{energy/time}{area} = \frac{power}{area}$$

Since E is proportional to A^2.

$$I \propto A^2$$

For isotropic medium, the wave propagates radially

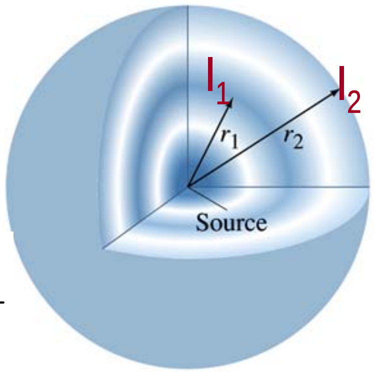
$$I = \frac{power}{area} = \frac{P}{4\pi r^2}$$

Ratio of intensities at two different radii is

$$\frac{I_2}{I_1} = \frac{P/4\pi r_2^2}{P/4\pi r_1^2} = \left(\frac{r_1}{r_2}\right)^2$$

Amplitude

$$\frac{A_2}{A_1} = \frac{r_1}{r_2}$$



Example 11 – 12

Earthquake intensity. If the intensity of an earthquake P wave 100km from the source is $1.0\times10^{7}W/m^{2}$, what is the intensity 400km from the source?

Since the intensity decreases as the square of the distance from the source,

$$\frac{I_2}{I_1} = \left(\frac{r_1}{r_2}\right)^2$$

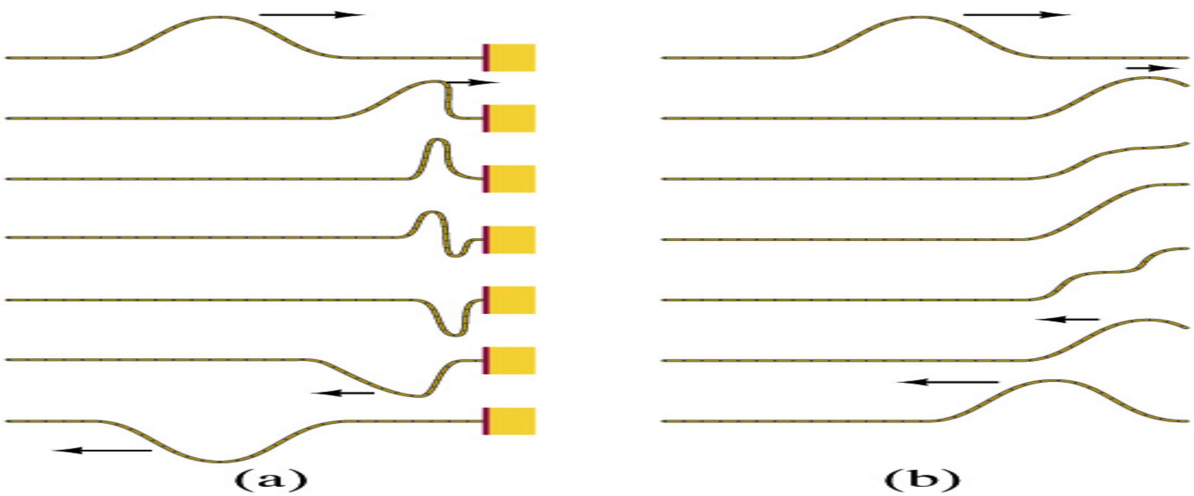
The intensity at 400km can be written in terms of the intensity at 100km

$$I_2 = \left(\frac{r_1}{r_2}\right)^2 I_1 = \left(\frac{100km}{400km}\right)^2 1.0\times10^{7} W/m^2 = 6.2\times10^{5} W/m^2$$

Reflection and Transmission

A pulse or a wave undergoes various changes when the medium it travels changes.

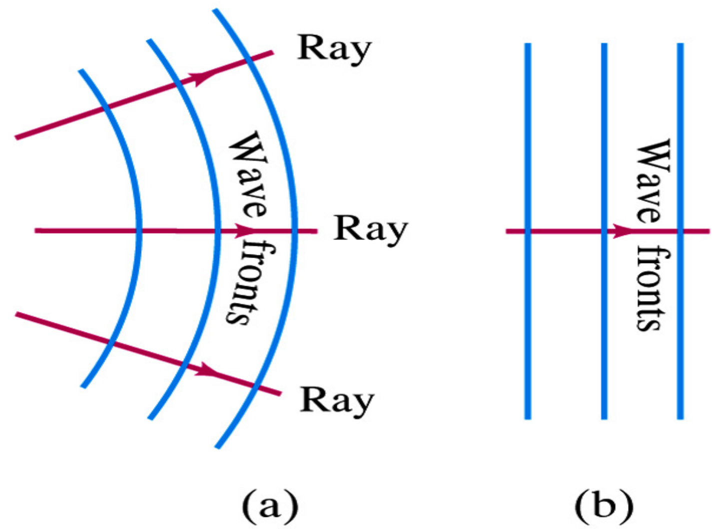
Depending on how rigid the support is, two radically different reflection patterns can be observed.



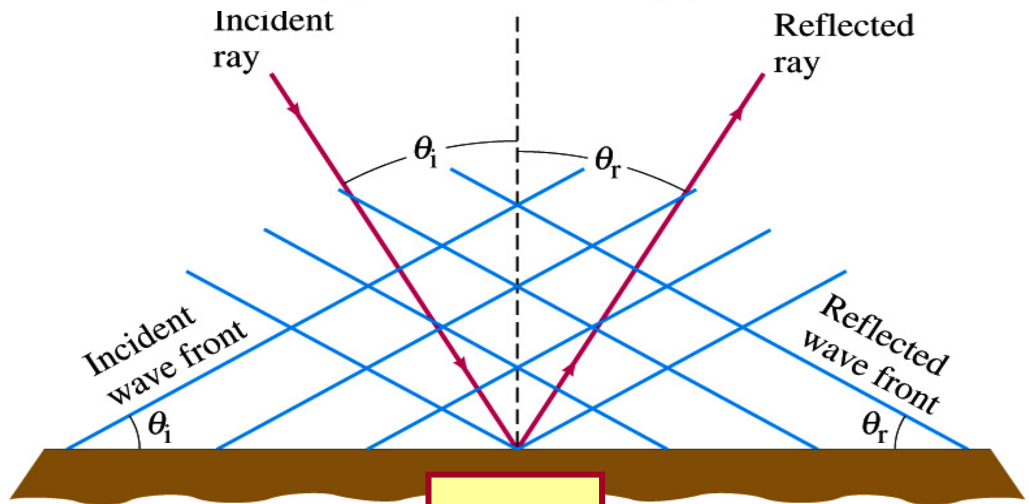
1. The support is rigidly fixed (a): The reflected pulse will be inverted to the original due to the force exerted on to the string by the support in reaction to the force on the support due to the pulse on the string.
2. The support is freely moving (b): The reflected pulse will maintain the original shape but moving in the reverse direction.

2 and 3 dimensional waves and the Law of Reflection

- Wave fronts: The whole width of wave crests
- Ray: A line drawn in the direction of motion, perpendicular to the wave fronts.
- Plane wave: The waves whose fronts are nearly straight



The Law of Reflection: The angle of reflection is the same as the angle of incidence.



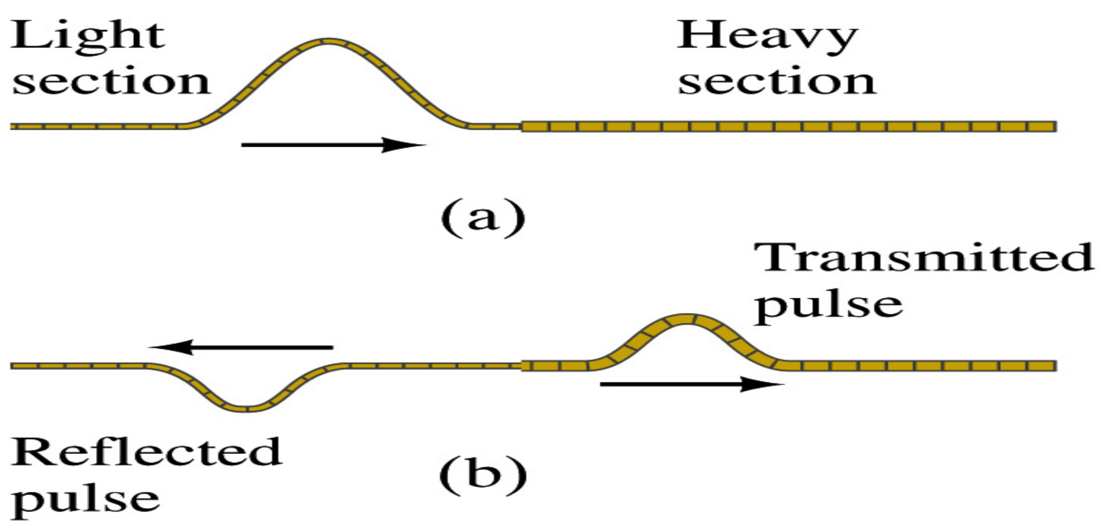
$\theta_i=\theta_r$

Transmission Through Different Media

If the boundary is intermediate between the previous two extremes, part of the pulse reflects, and the other undergoes transmission, passing through the boundary and propagating in the new medium.

When a wave pulse travels from medium A to B:

1. $v_A > v_B$ (or $\mu_A < \mu_B$), the pulse is inverted upon reflection
2. $v_A < v_B$ (or $\mu_A > \mu_B$), the pulse is not inverted upon reflection



Superposition Principle of Waves

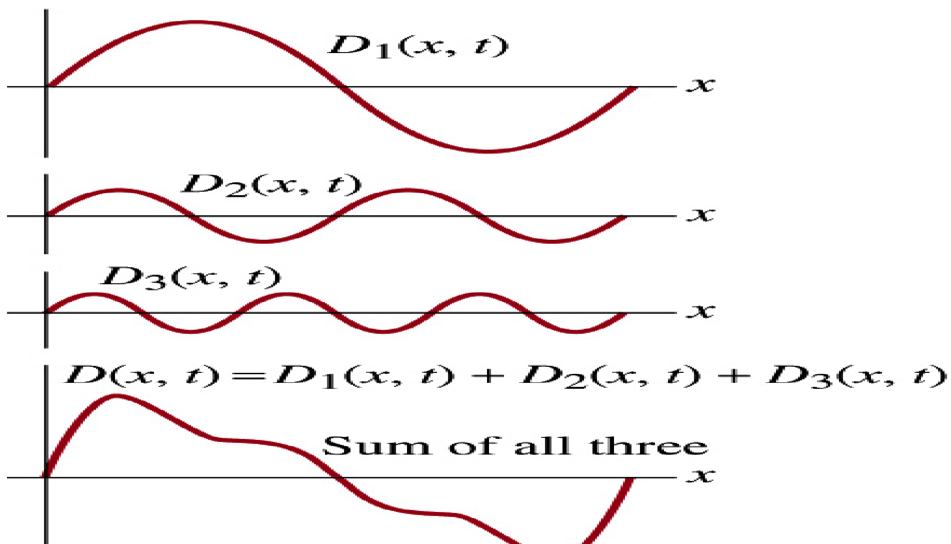
If two or more traveling waves are moving through a medium, the resultant wave function at any point is the algebraic sum of the wave functions of the individual waves.

Superposition Principle

The waves that follow this principle are called ***linear waves*** which in general have small amplitudes. The ones that don't are ***nonlinear waves*** with larger amplitudes.

Thus, one can write the resultant wave function as

$$y = y_1 + y_2 + \cdots + y_n = \sum_{i=1}^{n} y_i$$



Wave Interferences

Two traveling linear waves can pass through each other without being destroyed or altered.

What do you think will happen to the water waves when you throw two stones in the pond?

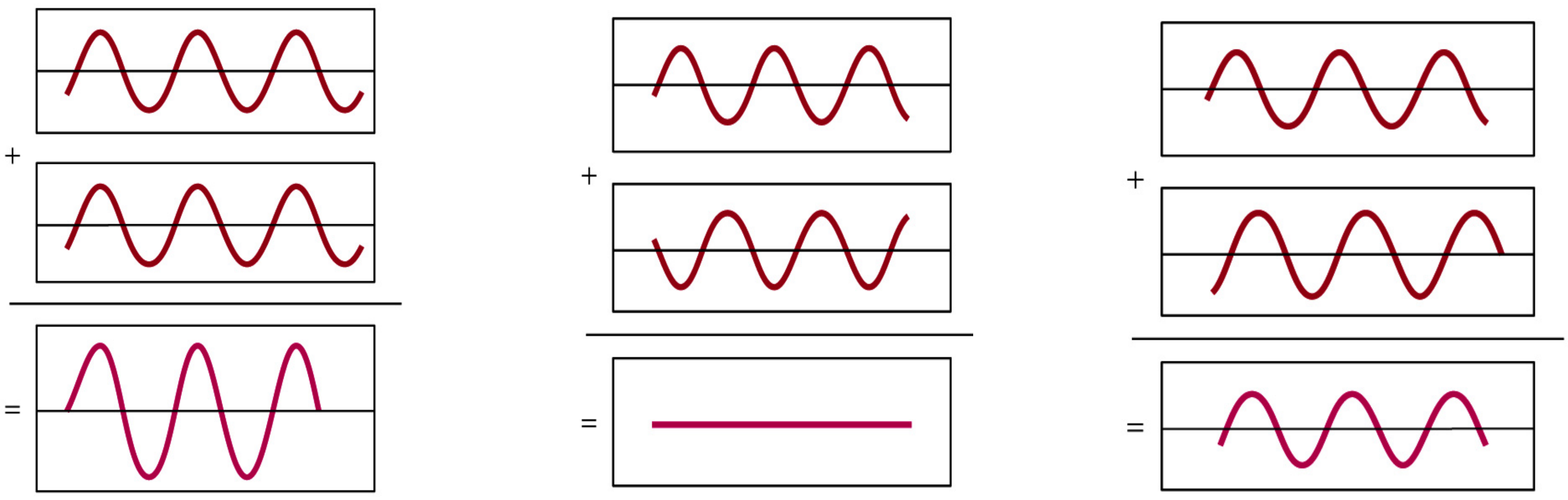
They will pass right through each other.

What happens to the waves at the point where they meet?

The shape of wave will change ➔ Interference

Constructive interference: The amplitude increases when the waves meet

Destructive interference: The amplitude decreases when the waves meet



In phase ➔ constructive

Out of phase by $\pi/2$ ➔ destructive

Out of phase not by $\pi/2$ ➔ Partially destructive