PHYS 3446 – Lecture #4

Monday, Jan. 31, 2005

Dr. Jae Yu

1. Lab Frame and Center of Mass Frame
2. Relativistic Treatment
3. Feynman Diagram
4. Quantum Treatment of Rutherford Scattering
5. Nuclear Phenomenology: Properties of Nuclei
6. A few measurements of differential cross sections

Announcements

- I still have eleven subscribed the distribution list.
- I really hate doing this but since this is the primary class communication tool, it is critical for all of you to subscribe.
  – I will assign -3 points to those of you not registered by the class Wednesday
- A test message will be sent out Wednesday
- Must take the radiation safety training!!!
  – Directly related to your lab score ➔ 15% of the total
- Homework extra credit: 5 points if done by the class after the assignment!

Scattering Cross Section

- For a central potential, measuring the yield as a function of θ, or differential cross section, is equivalent to measuring the entire effect of the scattering
- So what is the physical meaning of the differential cross section?

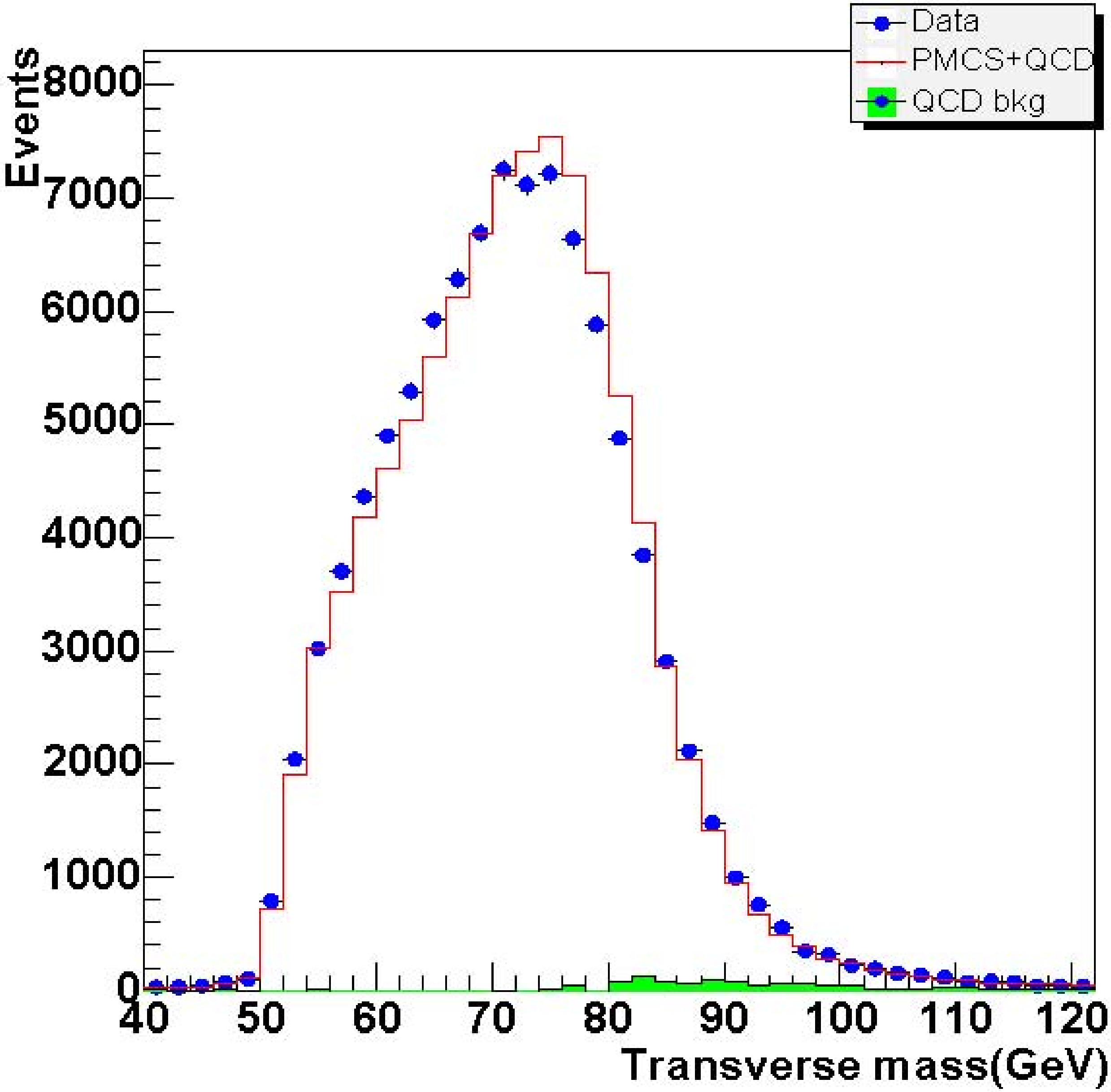
$\Rightarrow$ Measurement of yield as a function of specific experimental variable

$\Rightarrow$ This is equivalent to measuring the probability of certain process in a specific kinematic phase space

- Cross sections are measured in the unit of barns:

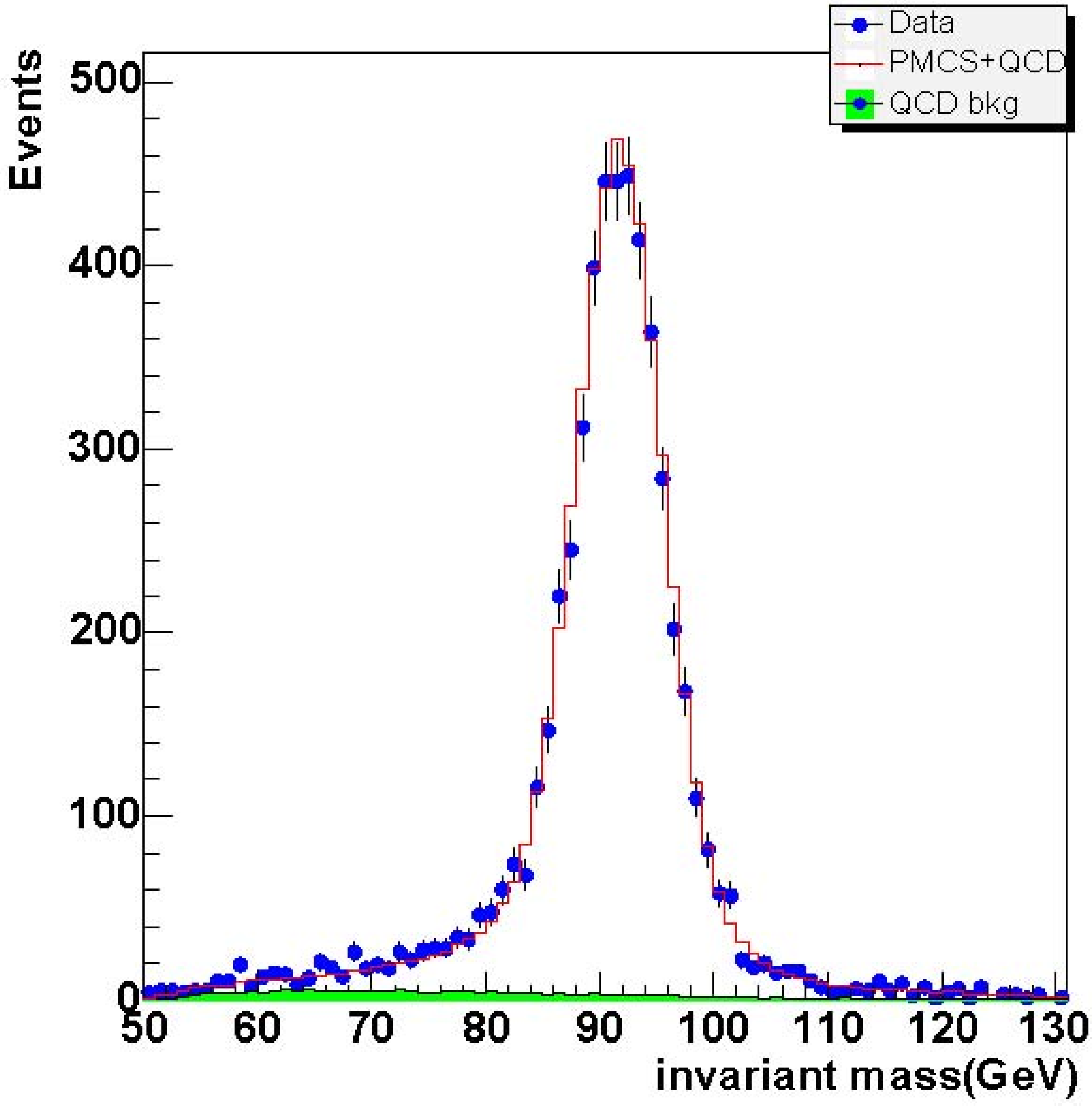
$$1\ \text{barn} = 10^{-24}\text{cm}^2$$

Example Cross Section: W($\rightarrow$eν) +X



- Transverse mass distribution of electrons in W+X events

Example Cross Section: W(→ee) +X

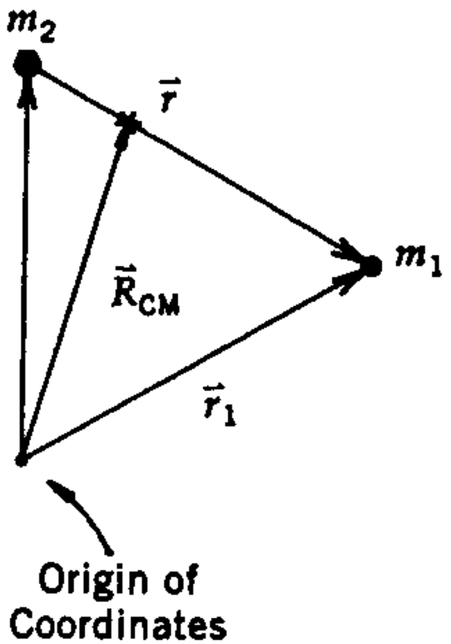


- Invariant mass distribution of electrons in Z+X events

Lab Frame and Center of Mass Frame

- We assumed that the target nuclei do not move through the collision
- In reality, they recoil as a result of scattering
- Sometimes we use two beams of particles for scattering experiments (target is moving)
- This situation could be complicated but..
- Could be simplified if the motion can be described in Center of Mass frame under a central potential

Lab Frame and CM Frame



$$\vec{r}_1 = r_1 \hat{r}_1$$
$$\vec{r}_2 = r_2 \hat{r}_2$$
$$\vec{r} = r \hat{r}$$
$$\vec{R}_{CM} = R_{CM} \hat{R}_{CM}$$

- The equations of motion can be written

$$m_1 \ddot{\vec{r}}_1 = -\vec{\nabla}_1 V\left(\left|\vec{r}_1 - \vec{r}_2\right|\right)$$
$$m_2 \ddot{\vec{r}}_2 = -\vec{\nabla}_2 V\left(\left|\vec{r}_1 - \vec{r}_2\right|\right)$$

where $\vec{\nabla}_i = \hat{r}_i \dfrac{\partial}{\partial r_i} + \dfrac{\hat{\theta}_i}{r_i}\dfrac{\partial}{\partial \theta_i} + \dfrac{\hat{\phi}_i}{r_i \sin\theta_i}\dfrac{\partial}{\partial \phi_i}; \quad i = 1, 2$

Since the potential depends only on relative separation of the particles, we redefine new variables

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \quad \text{and} \quad \vec{R}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

Lab Frame and CM Frame

- From the equations in previous slides

$$\boxed{\frac{m_1 m_2}{m_1+m_2}\ddot{\vec{r}} \equiv \mu\ddot{\vec{r}} = -\vec{\nabla}V\left(\left|\vec{r}\right|\right) = -\frac{\partial V\left(\left|\vec{r}\right|\right)}{\partial r}\hat{r}}$$

Reduced Mass

and $$\boxed{\left(m_1+m_2\right)\ddot{\vec{R}}_{CM} = M\ddot{\vec{R}}_{CM} = 0}$$

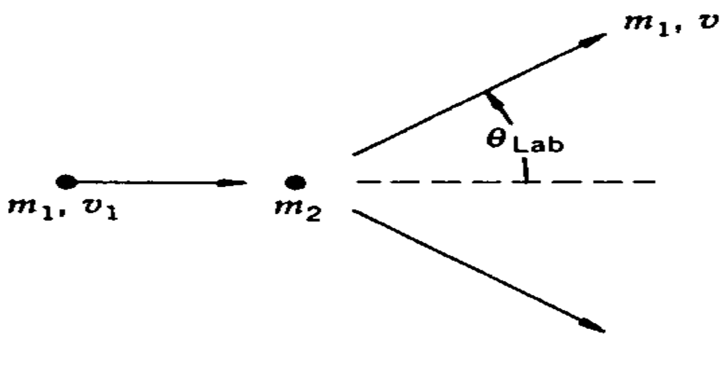
Thus $$\dot{\vec{R}}_{CM} = \text{constant}\ \hat{R}$$

- What do we learn from this exercise?
- For a central potential, the motion of the two particles can be decoupled when re-written in terms of
  - a relative coordinate
  - The coordinate of center of mass

Lab Frame and CM Frame

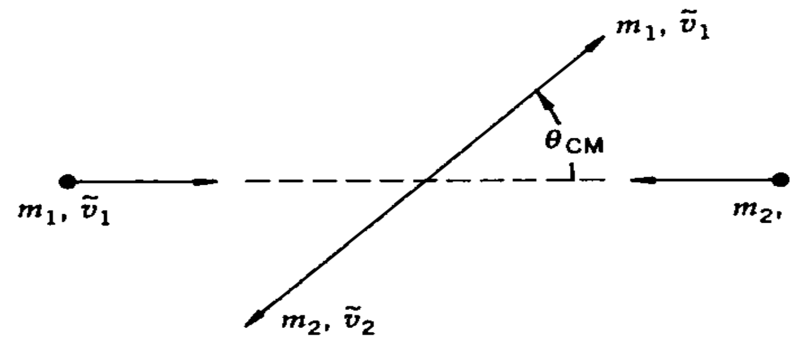
- The CM is moving at a constant speed in Lab frame independent of the form of the central potential
- The motion is as if that of a fictitious particle with reduced mass μ and coordinate r.
- In the frame where CM is stationary, the dynamics becomes equivalent to that of a single particle of mass μ scattering of a fixed, scattering center.
- Frequently we define the Center of Mass frame as the frame where the sum of the momenta of the interacting particles vanish.

Relationship of variables in Lab and CMS



Lab Frame

CMS



CM Frame

- The speed of CM is $$v_{CM} = \dot{R}_{CM} = \frac{m_1 v_1}{m_1 + m_2}$$

- Speeds of particles in CMS are

$$\tilde{v}_1 = v_1 - v_{CM} = \frac{m_2 v_1}{m_1 + m_2} \quad \text{and} \quad \tilde{v}_2 = v_{CM} = \frac{m_1 v_1}{m_1 + m_2}$$

- The momenta of the two particles are equal and opposite!!

Scattering angles in Lab and CMS

- θ_{CM} represents the changes in the direction of the relative position vector r as a result of the collision
- Thus, it must be identical to the scattering angle for the particle with the reduced mass, μ.
- Z components of the velocities of particle with m_1 in lab and CMS are:

$$v\cos\theta_{Lab} - v_{CM} = \tilde{v}_1 \cos\theta_{CM}$$

- The perpendicular components of the velocities are:

$$v\sin\theta_{Lab} = \tilde{v}_1 \sin\theta_{CM}$$

- Thus, the angles are related, for elastic scattering only, as:

$$\tan\theta_{Lab} = \frac{\sin\theta_{CM}}{\sin\theta_{CM} + v_{CM}/\tilde{v}_1} = \frac{\sin\theta_{CM}}{\sin\theta_{CM} + m_1/m_2} = \frac{\sin\theta_{CM}}{\sin\theta_{CM} + \zeta}$$

Differential cross sections in Lab and CMS

- The particles scatter at an angle θ_{Lab} into solid angle $d\Omega_{Lab}$ in lab scatters into θ_{CM} into solid angle $d\Omega_{CM}$ in CM.
- Since ϕ is invariant, $d\phi_{Lab} = d\phi_{CM}$.
  - Why?
    - ϕ is perpendicular to the direction of boost, thus is invariant.
- Thus, the differential cross section becomes:

$$\frac{d\sigma}{d\Omega_{Lab}}(\theta_{Lab})\sin\theta_{Lab}d\theta_{Lab} = \frac{d\sigma}{d\Omega_{CM}}(\theta_{CM})\sin\theta_{CM}d\theta_{CM}$$

rewrite

$$\frac{d\sigma}{d\Omega_{Lab}}(\theta_{Lab}) = \frac{d\sigma}{d\Omega_{CM}}(\theta_{CM})\frac{d(\cos\theta_{CM})}{d(\cos\theta_{Lab})}$$

Using Eq. 1.53

$$\boxed{\frac{d\sigma}{d\Omega_{Lab}}(\theta_{Lab}) = \frac{d\sigma}{d\Omega_{CM}}(\theta_{CM})\frac{\left(1+2\zeta\cos\theta_{CM}+\zeta^2\right)^{3/2}}{\left|1+\zeta\cos\theta_{CM}\right|}}$$

Using Eq. 1.53

Relativistic Variables

- Velocity of CM in the scattering of two particles with rest mass m_1 and m_2 is:

$$\frac{\vec{v}_{CM}}{c} = \vec{\beta}_{CM} = \frac{\left(\vec{P}_1 + \vec{P}_2\right)c}{E_1 + E_2}$$

- If m_1 is the mass of the projectile and m_2 is that of the target, for fixed target we obtain

$$\vec{\beta}_{CM} = \frac{\vec{P}_1 c}{E_1 + E_2} = \frac{\vec{P}_1 c}{\sqrt{P_1^2 c^2 + m_1^2 c^4} + m_2 c^2}$$

Relativistic Variables

- At very low energies where $m_1c^2 >> P_1c$, the velocity reduces to:

$$\vec{\beta}_{CM} = \frac{m_1 \vec{v}_1 c}{m_1 c^2 + m_2 c^2} = \frac{m_1 \vec{v}_1}{(m_1 + m_2)c}$$

- At very high energies where $m_1c^2 << P_1c$ and $m_2c^2 << P_1c$, the velocity can be written as:

$$\beta_{CM} = \left|\vec{\beta}_{CM}\right| = \frac{1}{\sqrt{1 + \left(\frac{m_1 c^2}{P_1 c}\right)^2 + \frac{m_2 c^2}{P_1 c}}} \approx 1 - \frac{m_2 c}{P_1} - \frac{1}{2}\left(\frac{m_1 c}{P_1}\right)^2$$

Relativistic Variables

- For high energies, if $m_1 \sim m_2$,

$$\beta_{CM} \approx \left(1 - \frac{m_2 c}{P_1}\right)^2$$

- γ_{CM} becomes:

$$\gamma_{CM} = \left(1 - \beta_{CM}^2\right)^{-1/2} \approx \left[\left(1 - \beta_{CM}\right)\left(1 + \beta_{CM}\right)\right]^{-1/2} \approx \left[2\left(\frac{m_2 c}{P_1}\right)\right]^{-1/2} = \sqrt{\frac{P_1}{2 m_2 c}}$$

- And

$$1 - \beta_{CM}^2 = \frac{m_1^2 c^4 + 2 E_1 m_2 c^2 + m_2^2 c^4}{\left(E_1 + m_2 c^2\right)^2}$$

- Thus γ_{CM} becomes

$$\gamma_{CM} = \left(1 - \beta_{CM}^2\right)^{-1/2} = \frac{E_1 + m_2 c^2}{\sqrt{m_1^2 c^4 + 2 E_1 m_2 c^2 + m_2^2 c^4}}$$

Invariant Scalar: s

Relativistic Variables

- The invariant scalar, s, is defined as:

$$s = \left(E_1 + E_2\right)^2 - \left(\vec{P}_1 + \vec{P}_2\right)^2 c^2$$
$$= m_1^2 c^4 + m_2^2 c^4 + 2E_1 m_2 c^2$$

- In the CMS frame

$$s = m_1^2 c^4 + m_2^2 c^4 + 2E_1 m_2 c^2$$
$$= \left(E_{1CM} + E_{2CM}\right)^2 - \left(\vec{P}_{1CM} + \vec{P}_{2CM}\right)^2 c^2$$
$$= \left(E_{1CM} + E_{2CM}\right)^2 = \left(E_{ToT}^{CM}\right)^2$$

- Thus, $\sqrt{s}$ represents the total available energy in the CMS

Useful Invariant Scalar Variables

- Another invariant scalar, t, the momentum transfer, is useful for scattering:

$$t = \left(E_1^f - E_1^i\right)^2 - \left(\vec{P}_1^f - \vec{P}_1^i\right)^2 c^2$$

- Since momentum and energy are conserved in all collisions, t can be expressed in terms of target variables

$$t = \left(E_2^f - E_2^i\right)^2 - \left(\vec{P}_2^f - \vec{P}_2^i\right)^2 c^2$$

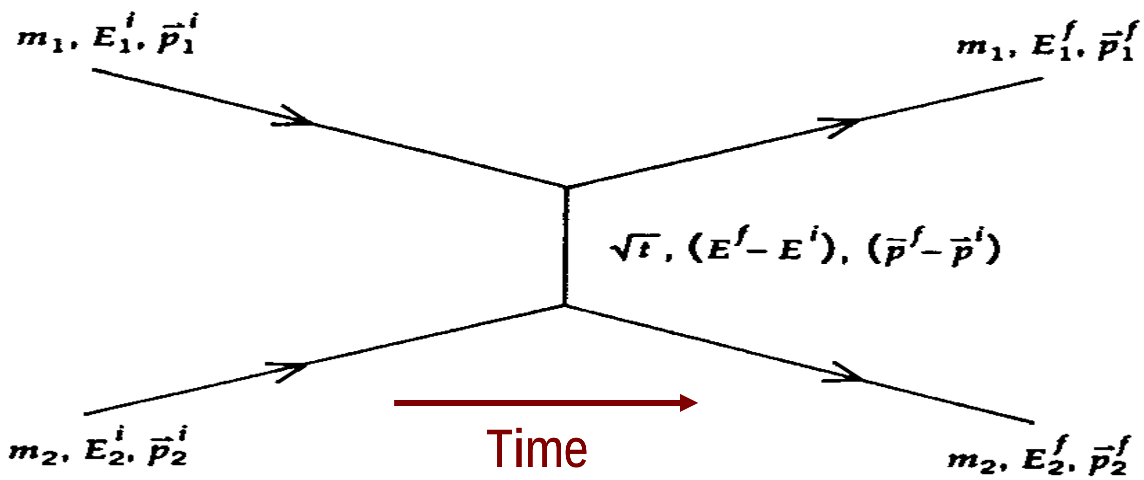
What does this variable look like?

- In CMS frame for an elastic scattering, where $P^i_{CM}=P^f_{CM}=P_{CM}$ and $E^i_{CM}=E^f_{CM}$:

$$t = -\left(P_{1CM}^{f\,2} + P_{1CM}^{i\,2} - 2\vec{P}_1^f \cdot \vec{P}_1^i\right)^2 c^2 = -2P_{CM}^2 c^2 \left(1 - \cos\theta_{CM}\right).$$

Feynman Diagram

- The variable t is always negative for an elastic scattering
- The variable t could be viewed as the square of the invariant mass of a particle with $E_2^f - E_2^i$ and $\vec{P}_2^f - \vec{P}_2^i$ exchanged in the scattering



t-channel diagram

- While the virtual particle cannot be detected in the scattering, the consequence of its exchange can be calculated and observed!!!

Useful Invariant Scalar Variables

- For convenience we define a variable q^2, $q^2c^2 = -t$
- In the lab frame $\vec{P}^{i}_{2Lab} = 0$, thus we obtain:

$$q^2c^2 = -\left[\left(E^{f}_{2Lab} - m_2c^2\right)^2 - \left(P^{f}_{2Lab}c\right)^2\right]$$

$$= 2m_2c^2\left(E^{f}_{2Lab} - m_2c^2\right) = 2m_2c^2T^{f}_{2Lab}$$

- In the non-relativistic limit: $T^{f}_{2Lab} \approx \frac{1}{2}m_2v_2^2$
- q^2 represents "hardness of the collision". Small θ_{CM} corresponds to small q^2.

Relativistic Scattering Angles in Lab and CMS

- For a relativistic scattering, the relationship between the scattering angles in Lab and CMS is:

$$\tan\theta_{Lab} = \frac{\tilde{\beta}\sin\theta_{CM}}{\gamma_{CM}\left(\tilde{\beta}\sin\theta_{CM} + \beta_{CM}\right)}$$

- For Rutherford scattering ($m=m_1<<m_2$, $v\sim v_0<<c$):

$$dq^2 = -2P^2 d\left(\cos\theta\right) = \frac{P^2 d\Omega}{\pi}$$

Resulting in a cross section

$$\frac{d\sigma}{dq^2} = \frac{4\pi\left(ZZ'e^2\right)^2}{v^2}\frac{1}{q^4}$$

- Divergence at $q^2\sim 0$, a characteristics of a Coulomb field
- There are distribution of q^2 in Rutherford scattering which falls off rapidly

Assignments

1. Derive the following equations:
   - Eqs. 1.51, 1.53, 1.55, 1.63, 1.71
   - Prove Eq. 1.45
   - Derive Eq. 1.51 from Eq. 1.71 in its non-relativistic limit
2. Compute the available CMS energy ($\sqrt{s}$) for
   - Fixed target experiment with masses m_1 and m_2 with incoming energy E_1.
   - Collider experiment with masses m_1 and m_2 with incoming energies E_1 and E_2.
3. Reading assignment: Section 1.7
4. End of chapter problems:
   - 1.1 and 1.7

- These assignments are due next Monday, Feb. 7.