PHYS 3446 – Lecture #16

Monday, Apr. 4, 2005
Dr. Jae Yu

- Symmetries
  - Why do we care about the symmetry?
  - Symmetry in Lagrangian formalism
  - Symmetries in quantum mechanical system
  - Isospin symmetry
  - Local gauge symmetry

Announcements

- 3rd Quiz this Wednesday, Apr. 6
  - Covers: Ch. 9 and 10.5
- Don't forget that you have another opportunity to do your past due homework at 85% of full if you submit the by Wed., Apr. 20
- Will have an individual mid-semester discussion this week

Quantum Numbers

- We've learned about various newly introduced quantum numbers as a patch work to explain experimental observations
  - Lepton numbers
  - Baryon numbers
  - Isospin
  - Strangeness
- Some of these numbers are conserved in certain situation but not in others
  - Very frustrating indeed….
- These are due to lack of quantitative description by an elegant theory

Why symmetry?

- Some of the quantum numbers are conserved in strong interactions but not in electromagnetic and weak interactions
  - Inherent reflection of underlying forces
- Understanding conservation or violation of quantum numbers in certain situations is important for formulating quantitative theoretical framework

Why symmetry?

- When does a quantum number conserved?
  - When there is an underlying symmetry in the system
  - When the quantum number is not affected (or is conserved) by (under) changes in the physical system
- Noether's theorem: If there is a conserved quantity associated with a physical system, there exists an underlying invariance or symmetry principle responsible for this conservation.
- Symmetries provide critical restrictions in formulating theories

Symmetries in Lagrangian Formalism

- Symmetry of a system is defined by any set of transformations that keep the equation of motion unchanged or invariant
- Equations of motion can be obtained through
  - Lagrangian formalism: L=T-V where the Equation of motion is what minimizes the lagrangian L under changes of coordinates
  - Hamiltonian formalism: H=T+V with the equation of motion that minimizes the Hamiltonian under changes of coordinates
- Both these formalisms can be used to discuss symmetries in non-relativistic (or classical cases), relativistic, and quantum mechanical systems

Symmetries in Lagrangian Formalism?

- Consider an isolated non-relativistic physical system of two particles interacting through a potential that only depends on the relative distance between them
- The total kinetic and potential energies of the system are: $T = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2}$ and $V = V\left(\vec{r}_1 - \vec{r}_2\right)$
- The equations of motion are then $\frac{d}{dt}\frac{\partial L_i}{\partial \dot{\vec{r}}} - \frac{\partial L_i}{\partial \vec{r}} = 0$

$$m_1 \ddot{\vec{r}}_1 = -\vec{\nabla}_1 V\left(\vec{r}_1 - \vec{r}_2\right) = -\frac{\partial}{\partial \vec{r}_1} V\left(\vec{r}_1 - \vec{r}_2\right)$$

$$m_2 \ddot{\vec{r}}_2 = -\vec{\nabla}_2 V\left(\vec{r}_1 - \vec{r}_2\right) = -\frac{\partial}{\partial \vec{r}_2} V\left(\vec{r}_1 - \vec{r}_2\right)$$

where $\frac{\partial}{\partial \vec{r}_i} V\left(\vec{r}_1 - \vec{r}_2\right) =$

$$= \hat{x}\frac{\partial}{\partial x_i}V + \hat{y}\frac{\partial}{\partial y_i}V + \hat{z}\frac{\partial}{\partial x_i}V$$

Symmetries in Lagrangian Formalism

- If we perform a linear translation of the origin of coordinate system by a constant vector $-\vec{a}$
  - The position vectors of the two particles become

$$\vec{r}_1 \rightarrow \vec{r}_1 - \vec{a} \qquad \vec{r}_2 \rightarrow \vec{r}_2 - \vec{a}$$

  - But the equation of motions do not change since $-\vec{a}$ is a constant vector
  - This is due to the invariance of the potential V under the translation

$$V' = V\left(\vec{r}\,'_1 - \vec{r}\,'_2\right) = V\left(\vec{r}_1 - \vec{a} - \vec{r}_2 + \vec{a}\right) = V\left(\vec{r}_1 - \vec{r}_2\right)$$

Symmetries in Lagrangian Formalism

- This means that the translation of the coordinate system for an isolated two particle system defines a symmetry of the system (remember Noether's theorem?)
- This particular physical system is invariant under spatial translation
- What is the consequence of this invariance?
  - From the form of the potential, the total force is

$$\vec{F}_{tot} = \vec{F}_1 + \vec{F}_2 = -\vec{\nabla}_1 V\left(\vec{r}_1 - \vec{r}_2\right) - \vec{\nabla}_2 V\left(\vec{r}_1 - \vec{r}_2\right) = 0$$

  - Since $\frac{\partial V}{\partial \vec{r}_1} = -\frac{\partial V}{\partial \vec{r}_2}$ Why?

Symmetries in Lagrangian Formalism

- What does this mean?
  - Total momentum of the system is invariant under spatial translation

$$\vec{F}_{tot} = \frac{d\vec{P}_{tot}}{dt} = 0$$

- In other words, the translational symmetry results in momentum conservation
- This holds for multi-particle, multi-variable system as well!!

Symmetries in Lagrangian Formalism

- For multi-particle system, using Lagrangian L=T-V the equations of motion can be generalized
$$\frac{d}{dt}\frac{\partial L_i}{\partial \dot{\vec{r}}}-\frac{\partial L_i}{\partial \vec{r}}=0$$
- By construction, $$\frac{\partial L_i}{\partial \dot{\vec{r}}}=\frac{\partial T_i}{\partial \dot{\vec{r}}}= m_i\dot{\vec{r}}=\vec{p}_i$$
- As previously discussed, for the system with a potential that depends on the relative distance between particles, lagrangian is independent of particulars of the individual coordinate $\frac{\partial L_i}{\partial r_m}=0$ and thus $\frac{d\vec{p}_i}{dt}=\frac{\partial L_i}{\partial r_i}=0$

Symmetries in Lagrangian Formalism

- The momentum p_i can expanded to other kind of momenta for the given spatial translation
  - Rotational translation: Angular momentum
  - Time translation: Energy
  - Rotation in isospin space: Isospin
- The equation $\frac{d\vec{p}_i}{dt} = \frac{\partial L_i}{\partial r_i} = 0$ says that if the Lagrangian of a physical system does not depend on specifics of a given coordinate, the conjugate momentum are conserved
- One can turn this around and state that if a Lagrangian does not depend on some particular coordinate, it must be invariant under translations of this coordinate.

Symmetries in Translation and Conserved quantities

- The translational symmetries of a physical system dgive invariance in the corresponding physical quantities
  - Symmetry under linear translation
    - Linear momentum conservation
  - Symmetry under spatial rotation
    - Angular momentum conservation
  - Symmetry under time translation
    - Energy conservation
  - Symmetry under isospin space rotation
    - Isospin conservation

Symmetry in Quantum Mechanics

- In quantum mechanics, any observable physical quantity corresponds to the expectation value of a Hermitian operator in a given quantum state
  - The expectation value is given as a product of wave function vectors about the physical quantity (operator)

$$\langle Q \rangle = \langle \psi | Q | \psi \rangle$$

  - Wave function ($|\psi\rangle$)is the probability distribution function of a quantum state at any given space-time coordinates
  - The observable is invariant or conserved if the operator Q commutes with Hamiltonian

Continuous Symmetry

- All symmetry transformations of a theory can be categorized in
  - Continuous symmetry: Symmetry under continuous transformation
    - Spatial translation
    - Time translation
    - Rotation
  - Discrete symmetry: Symmetry under discrete transformation
    - Transformation in discrete quantum mechanical system

Isospin

- If there is isospin symmetry, proton (isospin up, I_3= ½) and neutron (isospin down, I_3= -½) are indistinguishable
- Lets define a new neutron and proton states as some linear combination of the proton, $|p\rangle$, and neutron, $|n\rangle$, wave functions
- Then a finite rotation of the vectors in isospin space by an arbitrary angle θ about an isospin axis leads to a new set of transformed vectors

$$|p'\rangle = \cos\frac{\theta}{2}|p\rangle - \sin\frac{\theta}{2}|n\rangle$$

$$|n'\rangle = \sin\frac{\theta}{2}|p\rangle + \cos\frac{\theta}{2}|n\rangle$$

Isospin

- What does the isospin invariance mean to nucleon-nucleon interaction?
- Two nucleon quantum state can be written in the following four combinations of quantum states
  - Proton on proton (I_3=+1) $|\psi_1\rangle = |pp\rangle$
  - Neutron on neutron (I_3=-1) $|\psi_2\rangle = |nn\rangle$
  - Proton on neutron or neutron on proton for both symmetric or anti-symmetiric (I_3=0)

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\left(|pn\rangle + |np\rangle\right) \quad |\psi_4\rangle = \frac{1}{\sqrt{2}}\left(|pn\rangle - |np\rangle\right)$$

Isospin Tranformation

- For I_3=+1 wave function:

$$|\psi_1'\rangle = \left|\left(\cos\frac{\theta}{2}p - \sin\frac{\theta}{2}n\right)\left(\cos\frac{\theta}{2}p - \sin\frac{\theta}{2}n\right)\right\rangle = \cos^2\frac{\theta}{2}|pp\rangle - \cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(|pn\rangle + |np\rangle\right) + \sin^2\frac{\theta}{2}|nn\rangle$$

$$= \cos^2\frac{\theta}{2}|\psi_1\rangle - \sqrt{2}\cos\frac{\theta}{2}\sin\frac{\theta}{2}|\psi_3\rangle + \sin^2\frac{\theta}{2}|\psi_2\rangle$$

- For I_3=0 anti-symmetric wave function

$$|\psi_4'\rangle = \frac{1}{\sqrt{2}}\left(\left|\left(\cos\frac{\theta}{2}p - \sin\frac{\theta}{2}n\right)\left(\sin\frac{\theta}{2}p + \cos\frac{\theta}{2}n\right)\right\rangle - \left|\left(\cos\frac{\theta}{2}p - \sin\frac{\theta}{2}n\right)\left(\sin\frac{\theta}{2}p + \cos\frac{\theta}{2}n\right)\right\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)\left(|pn\rangle - |np\rangle\right) = |\psi_4\rangle$$

  - This state is totally insensitive to isospin rotation➔ singlet combination of isospins (total isospin 0 state)

Isospin Tranformation

- The other three states corresponds to three possible projection state of the total isospin =1 state (triplet state)
- Thus, any two nucleon system can be in a singlet or a triplet state
- If there is isospin symmetry in strong interaction all these states are indistinguishable

Assignments

1. Construct the Lagrangian for an isolated, two particle system under a potential that depends only on the relative distance between the particles and show that the equations of motion from $\frac{d}{dt}\frac{\partial L_i}{\partial \dot{\vec{r}}} - \frac{\partial L_i}{\partial \vec{r}} = 0$ are

$$m_1\ddot{\vec{r}}_1 = -\vec{\nabla}_1 V(\vec{r}_1 - \vec{r}_2) = -\frac{\partial}{\partial \vec{r}_1} V(\vec{r}_1 - \vec{r}_2)$$

$$m_2\ddot{\vec{r}}_2 = -\vec{\nabla}_2 V(\vec{r}_1 - \vec{r}_2) = -\frac{\partial}{\partial \vec{r}_2} V(\vec{r}_1 - \vec{r}_2)$$

2. Prove that if $\psi(\vec{r})$ is a solution for the Schrodinger equation $H\psi(\vec{r}) = \left(-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{r})\right)\psi(\vec{r}) = E\psi(\vec{r})$, then $e^{i\alpha}\psi(\vec{r})$ is also a solution for it.
3. Due for this is next Monday, Apr. 11