1. Energy of the Simple Harmonic Oscillator
2. Simple Pendulum
3. Other Types of Pendulum
4. Damped Oscillation

Today’s homework is homework #19 due 12:00pm, Wednesday, Nov. 27!!
Announcements

• Evaluation today
• We have classes next week, both Monday and Wednesday
• Remember the Term Exam on Monday, Dec. 9 in the class
Equation of Simple Harmonic Motion

The solution for the 2nd order differential equation

\[ \frac{d^2x}{dt^2} = -\frac{k}{m} x \]

Generalized expression of a simple harmonic motion

Let's think about the meaning of this equation of motion

What happens when \( t=0 \) and \( \phi=0 \)?

\[ x = A \cos(0 + 0) = A \]

What is \( \phi \) if \( x \) is not \( A \) at \( t=0 \)?

\[ x = A \cos(\phi) = x' \]

\[ \phi = \cos^{-1}(x') \]

An oscillation is fully characterized by its:

- Amplitude
- Period or frequency
- Phase constant

What are the maximum/minimum possible values of \( x \)?

\( A/-A \)
More on Equation of Simple Harmonic Motion

What is the time for full cycle of oscillation?

Since after a full cycle the position must be the same
\[ x = A \cos(\omega (t + T) + \phi) = A \cos(\omega t + 2\pi + \phi) \]

One of the properties of an oscillatory motion

The period
\[ T = \frac{2\pi}{\omega} \]

How many full cycles of oscillation does this undergo per unit time?

One of the properties of an oscillatory motion

Frequency
\[ f = \frac{1}{T} = \frac{\omega}{2\pi} \]

What is the unit?
1/s=Hz

Let’s now think about the object’s speed and acceleration.

Speed at any given time
\[ v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi) \]

Max speed
\[ v_{\text{max}} = \omega A \]

Acceleration at any given time
\[ a = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \phi) = -\omega^2 x \]

Max acceleration
\[ a_{\text{max}} = \omega^2 A \]

What do we learn about acceleration?

Acceleration is reverse direction to displacement

Acceleration and speed are \( \pi/2 \) off phase:

When \( v \) is maximum, \( a \) is at its minimum

Wednesday, Nov. 20, 2002  
PHYS 1443-003, Fall 2002  
Dr. Jaehoon Yu
Energy of the Simple Harmonic Oscillator

How do you think the mechanical energy of the harmonic oscillator look without friction?

Kinetic energy of a harmonic oscillator is

\[ KE = \frac{1}{2} mv^2 = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi) \]

The elastic potential energy stored in the spring

\[ PE = \frac{1}{2} kx^2 = \frac{1}{2} kA^2 \cos^2(\omega t + \phi) \]

Therefore the total mechanical energy of the harmonic oscillator is

\[ E = KE + PE = \frac{1}{2} \left[ m\omega^2 A^2 \sin^2(\omega t + \phi) + kA^2 \cos^2(\omega t + \phi) \right] \]

Since \( \omega = \sqrt{\frac{k}{m}} \)

\[ E = KE + PE = \frac{1}{2} \left[ kA^2 \sin^2(\omega t + \phi) + kA^2 \cos^2(\omega t + \phi) \right] = \frac{1}{2} kA^2 \]

Total mechanical energy of a simple harmonic oscillator is a constant of a motion and is proportional to the square of the amplitude

Maximum KE is when PE=0

\[ KE_{\text{max}} = \frac{1}{2} m v_{\text{max}}^2 = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi) = \frac{1}{2} m\omega^2 A^2 = \frac{1}{2} kA^2 \]

One can obtain speed

\[ v = \pm \sqrt{\frac{k}{m}} (A^2 - x^2) = \pm \omega \sqrt{A^2 - x^2} \]
Example 13.4

A 0.500 kg cube connected to a light spring for which the force constant is 20.0 N/m oscillates on a horizontal, frictionless track. a) Calculate the total energy of the system and the maximum speed of the cube if the amplitude of the motion is 3.00 cm.

From the problem statement, $A$ and $k$ are

$$A = 3.00 \text{cm} = 0.03 \text{m}$$
$$k = 20.0 \text{N/m}$$

The total energy of the cube is

$$E = KE + PE = \frac{1}{2} kA^2 = 9.00 \times 10^{-3} \text{J}$$

Maximum speed occurs when kinetic energy is the same as the total energy

$$v_{\text{max}} = A \sqrt{\frac{k}{m}} = 0.03 \sqrt{\frac{20.0}{0.500}} = 0.190 \text{m/s}$$

b) What is the velocity of the cube when the displacement is 2.00 cm.

The velocity at any given displacement is

$$v = \sqrt{\frac{k}{m}(A^2 - x^2)} = \sqrt{20.0 \cdot (0.03^2 - 0.02^2) / 0.500} = 0.141 \text{m/s}$$

c) Compute the kinetic and potential energies of the system when the displacement is 2.00 cm.

Kinetic energy, $KE$

$$KE = \frac{1}{2} mv^2 = 0.500 \times (0.141)^2 = 4.97 \times 10^{-3} \text{J}$$

Potential energy, $PE$

$$PE = \frac{1}{2} kx^2 = 20.0 \times (0.02)^2 = 4.00 \times 10^{-3} \text{J}$$
The Pendulum

A simple pendulum also performs periodic motion.

The net force exerted on the bob is

\[ \sum F_r = T - mg \cos \theta_A = 0 \]

\[ \sum F_i = -mg \sin \theta_A = ma = m \frac{d^2 s}{dt^2} \]

Since the arc length, \( s \), is

\[ s = L \theta_A \]

Again became a second degree differential equation, satisfying conditions for simple harmonic motion

If \( \theta \) is very small, \( \sin \theta \sim \theta \)

\[ \frac{d^2 \theta}{dt^2} = -\frac{g}{L} \theta = -\omega^2 \theta \]

giving angular frequency

\[ \omega = \sqrt{\frac{g}{L}} \]

The period for this motion is

\[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L}{g}} \]

The period only depends on the length of the string and the gravitational acceleration.
Example 13.5

Christian Huygens (1629-1695), the greatest clock maker in history, suggested that an international unit of length could be defined as the length of a simple pendulum having a period of exactly 1s. How much shorter would our length unit be had this suggestion been followed?

Since the period of a simple pendulum motion is

\[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L}{g}} \]

The length of the pendulum in terms of T is

\[ L = \frac{T^2 g}{4\pi^2} \]

Thus the length of the pendulum when T=1s is

\[ L = \frac{T^2 g}{4\pi^2} = \frac{1 \times 9.8}{4\pi^2} = 0.248 \text{ m} \]

Therefore the difference in length with respect to the current definition of 1m is

\[ \Delta L = 1 - L = 1 - 0.248 = 0.752 \text{ m} \]
Physical Pendulum

Physical pendulum is an object that oscillates about a fixed axis which does not go through the object’s center of mass.

Consider a rigid body pivoted at a point O that is a distance d from the CM.

The magnitude of the net torque provided by the gravity is

\[ \sum \tau = -mgd \sin \theta \]

Then

\[ \sum \tau = I \alpha = I \frac{d^2\theta}{dt^2} = -mgd \sin \theta \]

Therefore, one can rewrite

\[ \frac{d^2\theta}{dt^2} = -\frac{mgd}{I} \sin \theta \approx -\left(\frac{mgd}{I}\right) \theta = -\omega^2 \theta \]

Thus, the angular frequency \( \omega \) is

\[ \omega = \sqrt{\frac{mgd}{I}} \]

And the period for this motion is

\[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{mgd}} \]

By measuring the period of physical pendulum, one can measure moment of inertia.

Does this work for simple pendulum?
Example 13.6

A uniform rod of mass $M$ and length $L$ is pivoted about one end and oscillates in a vertical plane. Find the period of oscillation if the amplitude of the motion is small.

Moment of inertia of a uniform rod, rotating about the axis at one end is

$$I = \frac{1}{3} ML^2$$

The distance $d$ from the pivot to the CM is $L/2$, therefore the period of this physical pendulum is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{Mgd}} = 2\pi \sqrt{\frac{2ML^2}{3MgL}} = 2\pi \sqrt{\frac{2L}{3g}}$$

Calculate the period of a meter stick that is pivot about one end and is oscillating in a vertical plane.

Since $L=1m$, the period is

$$T = 2\pi \sqrt{\frac{2L}{3g}} = 2\pi \sqrt{\frac{2}{3 \cdot 9.8}} = 1.64s$$

So the frequency is

$$f = \frac{1}{T} = 0.61s^{-1}$$
Torsional Pendulum

When a rigid body is suspended by a wire to a fixed support at the top and the body is twisted through some small angle $\theta$, the twisted wire can exert a restoring torque on the body that is proportional to the angular displacement.

The torque acting on the body due to the wire is

$$\tau = -\kappa \theta$$

$\kappa$ is the torsion constant of the wire

Applying the Newton's second law of rotational motion

$$\sum \tau = I \alpha = I \frac{d^2 \theta}{dt^2} = -\kappa \theta$$

Then, again the equation becomes

$$\frac{d^2 \theta}{dt^2} = -\left(\frac{\kappa}{I}\right) \theta = -\omega^2 \theta$$

Thus, the angular frequency $\omega$ is

$$\omega = \sqrt{\frac{\kappa}{I}}$$

And the period for this motion is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{\kappa}}$$

This result works as long as the elastic limit of the wire is not exceeded.